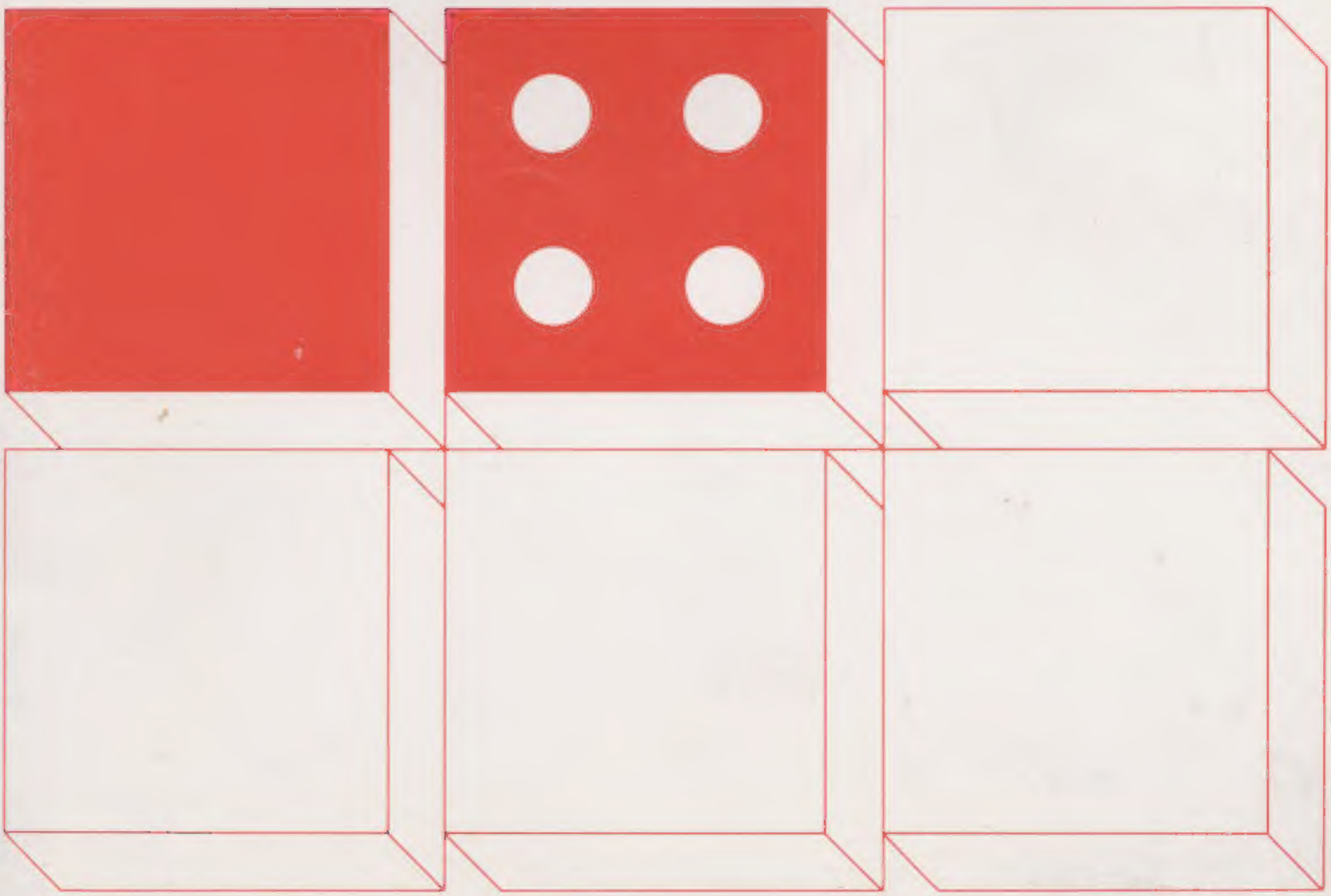
THE OPEN UNIVERSITY
Mathematics Foundation Course
Block II Functions and Numbers
Unit 4

FUNCTIONS AND LIMITS



THE OPEN UNIVERSITY

Mathematics: A Foundation Course

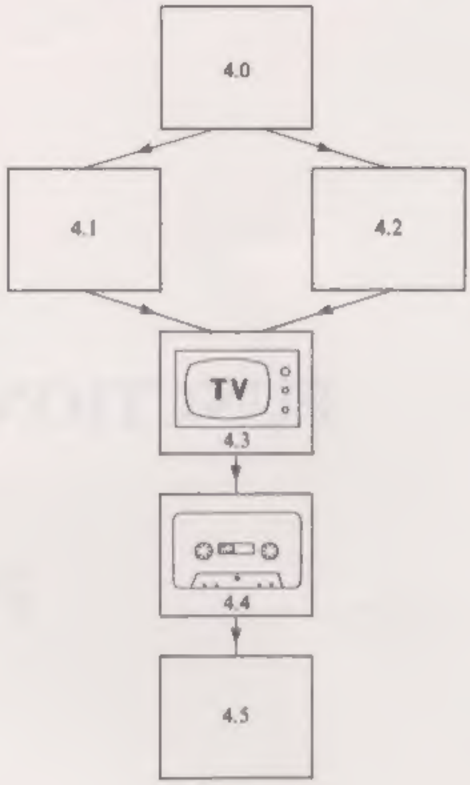
BLOCK II FUNCTIONS AND NUMBERS

Unit 4 Functions and Limits

Prepared by the Course Team

Contents

Dependence of Sections



Study Notes

The sections are intended to be studied in numerical order. Sections 4.1 and 4.2 are independent of each other and may be studied in either order; however, both 4.1 and 4.2 should be completed before tackling the TV section, Section 4.3.

If you are short of time, a quick reading of Section 4.5 will suffice for tackling the TMA; you could then return for a second look when time permits.

The Open University, Walton Hall, Milton Keynes.
First published 1978. New edition 1982. Reprinted 1985, 1987, 1988, 1989.

Designed by the Graphic Design Group of the Open University.
Typeset by Speedlith Photo Litho Limited, Longford Trading Estate, Manchester M32 0JT.
Printed by Hobbs the Printers Ltd, Second Avenue, Millbrook, Southampton SO9 2UZ.
This text forms part of the correspondence element of an Open University Foundation Course.
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2.5

4.0 INTRODUCTION

This unit sets out to answer two questions. The first, arising from Block I, *Unit 1*, is the problem of how to predict the success or failure of formula iteration,

$$x_{n+1} = f(x_n),$$

from knowledge of the function f. The second, arising from *Unit 1* of this block, is the problem of detecting whether or not a curve has horizontal tangents. We shall need the language and techniques of inequalities to investigate these two problems and we shall make use of sketch graphs of various functions to illustrate the ideas.

In Section 4.1 we recap on formula iteration and describe, more precisely than before, what we mean by a "successful" iteration.

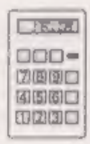
Section 4.2, which is independent of Section 4.1, investigates the problem of finding tangents to graphs and develops a technique for finding slopes of tangents. The TV section, Section 4.3, shows how knowledge of the behaviour of tangents enables you to predict the behaviour of formula iteration.

Section 4.4 aims to provide practice at finding slopes of tangents to the graphs of functions that we have been discussing previously in this block. Finally, in Section 4.5, we look at some useful language and notation for describing the methods used in this unit.

4.1 FORMULA ITERATION REVISITED

We begin this section by reminding you of the method of formula iteration, $x_{n+1} = f(x_n)$, used in Block I, *Unit 1*, to solve equations. We shall also introduce a graphical representation of iteration which will help to explain why some iterations converge and some do not.

Although we can solve quadratic equations by other methods, we use them to illustrate formula iteration.



Problem 4.1.1

(i) Use the iteration rule

$$x_{n+1} = x_n^2 - 4x_n + 6$$

to write down, to two decimal places, $x_2, x_3, \ldots, x_7$, starting from

(a) $x_1 = 3.5$; (b) $x_1 = 1.5$.

(ii) For which equation have you an approximate solution in part (i)(b)?

(iii) Find both solutions of the equation in the solution to part (ii).

We ask you to write down your answers to two decimal places; however, you should carry out the calculation using full accuracy, storing the current x_n value in memory.

Solution 4.1.1 illustrates two of the sorts of behaviour of formula iteration that you have already met. Starting with $x_1 = 3.5$, the iteration "blew-up", i.e. it *diverged*; starting with $x_1 = 1.5$, the iteration *converged* to the solution 2. However, the equation

To the accuracy of our calculator, $x_6 = x_7 = 2$.

$$x = x^2 - 4x + 6$$

has two solutions, 2 and 3, and neither of our starting values gave an iteration converging to 3. In order to stand a chance of converging to 3 with this iteration rule, it seems reasonable to take x_1 close to 3.

Problem 4.1.2

Investigate the behaviour of the iteration using the rule

$$x_{n+1} = x_n^2 - 4x_n + 6$$

for (i) $x_1 = 2.9$; (ii) $x_1 = 3.1$.

In fact, for the iteration rule $x_{n+1} = x_n^2 - 4x_n + 6$, any starting value near to 2 will give an iteration converging to 2 but no starting value, however close to 3, will give an iteration converging to 3. To help explain these differences in behaviour, it is useful to have a pictorial representation of formula iteration.

Of course, a starting value $x_1 = 3$ will give $x_2 = x_3 \ldots = 3$.

The solutions of the equation

$$x = x^2 - 4x + 6$$

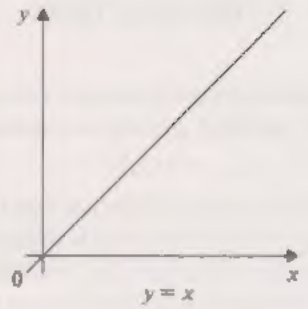
occur when the value of x and the value of $x^2 - 4x + 6$ are equal. So, if we plot the graphs of

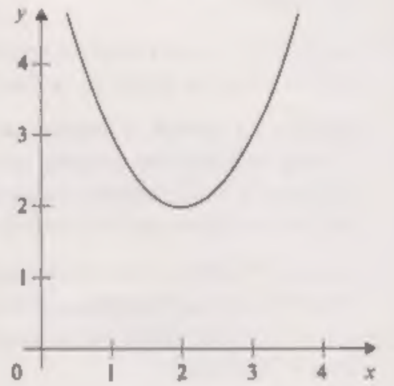
$$y = x$$

and

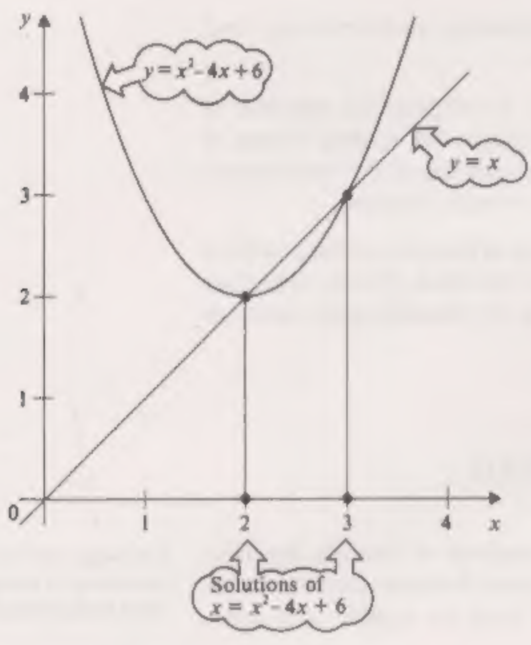
$$y = x^2 - 4x + 6 \quad (=(x-2)^2 + 2 \text{ in completed square form})$$

then the solutions of the equation $x = x^2 - 4x + 6$ will be the x-values where these graphs intersect.





$y = x^2 - 4x + 6$
i.e. $y = (x-2)^2 + 2$



We now use the above diagram to illustrate what happened in the convergent iteration of Solution 4.1.1(i)(b).

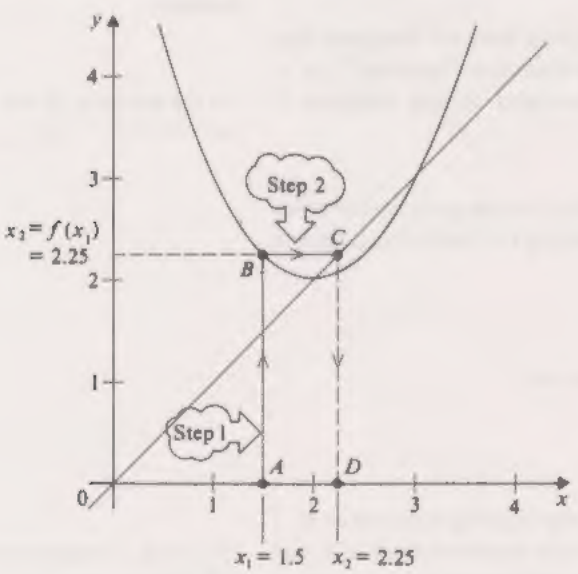
We started with $x_1 = 1.5$, and calculated

$$x_2 = f(x_1)$$

where, in this case,

$$f(x) = x^2 - 4x + 6.$$

This is illustrated by the vertical line AB in the diagram below.



Corresponding to $x_1 = 1.5$ at A on the x-axis, we obtain the point B whose coordinates are (1.5, 2.25), i.e.

$$(x_1, f(x_1)) = (x_1, x_2).$$

Now, all points on the horizontal line through B have the same y coordinate, namely x_2, and so C which is on the line $y = x$ has coordinates (x_2, x_2). Thus the point D on the x-axis corresponds to $x_2 = 2.25$. Summing up, we have moved from x_1 on the x-axis to x_2 on the x-axis by two steps:

Step 1: move vertically to the curve $y = f(x)$;
Step 2: move horizontally to the line $y = x$.

In this case, $f(x) = x^2 - 4x + 6$.

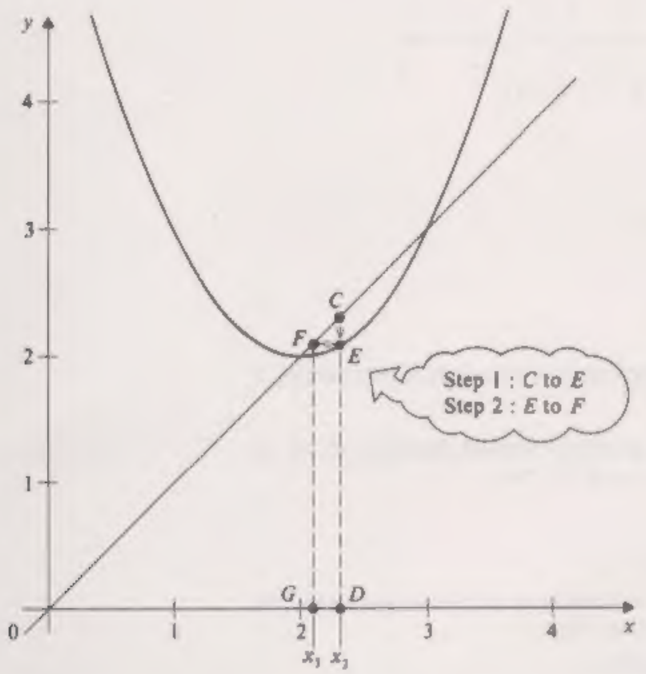
If we repeat these steps but start with x_2, we obtain

$$(x_2, f(x_2)) = (x_2, x_3) \quad (= (2.25, 2.0625) \text{ in this case})$$

on the curve at E and then

$$(x_3, x_3)$$

on the line $y = x$ at F.



Repeating these two steps will always take us from the current approximation, x_n, to the next, x_{n+1}. Graphically, starting with x_1 on the x-axis, we obtained a sequence of points, on the line $y = x$, whose coordinates are

(x_2, x_2) (C on the diagram),
(x_3, x_3) (F on the diagram),
(x_4, x_4),
etc.

As these points move towards (2,2), the intersection of the graphs, the sequence of numbers

$$x_1, x_2, x_3, \ldots$$

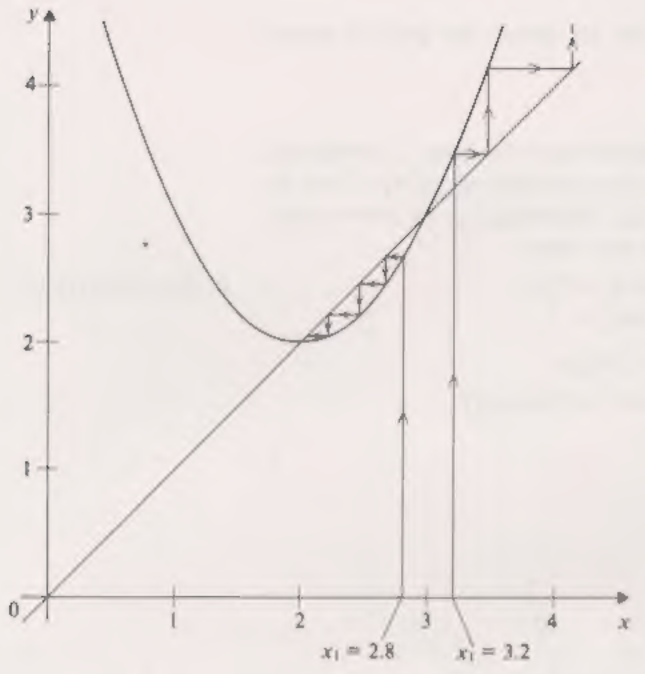
corresponding to the points A, D, G,... converges to the solution 2.

When we draw the same type of diagram but start with x_1 near 3, we see that these iterations do not converge to 3.

Vertically to the curve, horizontally to $y = x$.

For $x_1 = 3.2$, the corresponding sequence diverges; from the diagram overleaf, it is fairly clear that any starting point to the right of 3 will give a divergent sequence. For $x_1 = 2.8$, the corresponding sequence converges, but to 2 not 3; if you experiment graphically, you should be able to convince yourself that *any* starting value strictly between 2 and 3 will give a sequence converging to 2.

i.e. any starting value x_1 belonging to the open interval]2, 3[.



Problem 4.1.3

(i) Sketch, on the same axes, the graphs of

$$y = x$$

and

$$y = -x^2 + 4x - \tfrac{7}{4}.$$

(ii) What equation is satisfied by the x coordinates of the points of intersection of these two graphs?

(iii) Use the solution to part (i), and the graphical method described above, to predict the behaviour of the iteration using the rule

$$x_{n+1} = -x_n^2 + 4x_n - \tfrac{7}{4},$$

starting with (a) $x_1 = 2$; (b) $x_1 = 0.6$.

(iv) Use the solution to part (i) to predict the behaviour of the iteration using the rule $x_{n+1} = -x_n^2 + 4x_n - \frac{7}{4}$ for a starting value

(a) just to the left of the smaller solution;
(b) between the two solutions;
(c) just to the right of the larger solution.

Based on the above examples, it is reasonable to conjecture that the behaviour of the iteration using the rule

$$x_{n+1} = f(x_n)$$

depends on the shape of the graph of

$$y = f(x)$$

near the solutions of the equation

$$x = f(x).$$

Before we go any further, we ought to be a little more precise about what we mean by the sequence of approximations $x_1, x_2, \ldots, x_n, \ldots$ converging to the solution α of the corresponding equation.

The numbers

$$\alpha - x_1, \alpha - x_2, \alpha - x_3, \ldots$$

represent how good the approximations

$$x_1, x_2, x_3 \ldots$$

are to the solution α. So what we require is that the differences $\alpha - x_n$ tend to zero as n gets larger.

In the last section of this unit, we shall be a little more precise about the idea of "tending to zero as n gets larger".

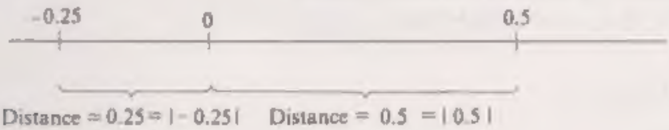
In Solution 4.1.1(i)(b), we obtained the approximations

$$x_1 = 1.5,\ x_2 = 2.25,\ x_3 = 2.0625, \ldots$$

to the solution $\alpha = 2$. The differences $\alpha - x_n$ are

$$0.5,\ -0.25,\ -0.0625, \ldots .$$

Now, x_2 $(=2.25)$ is a better approximation to α $(=2)$ because the difference, -0.25, is closer to zero than is 0.5.



What we are interested in is comparing the distances of these successive differences from 0, i.e. the moduli of the differences. To repeat, x_2 is better than x_1 as an approximation to α because

$$|-0.25| < |0.5|,$$

i.e.

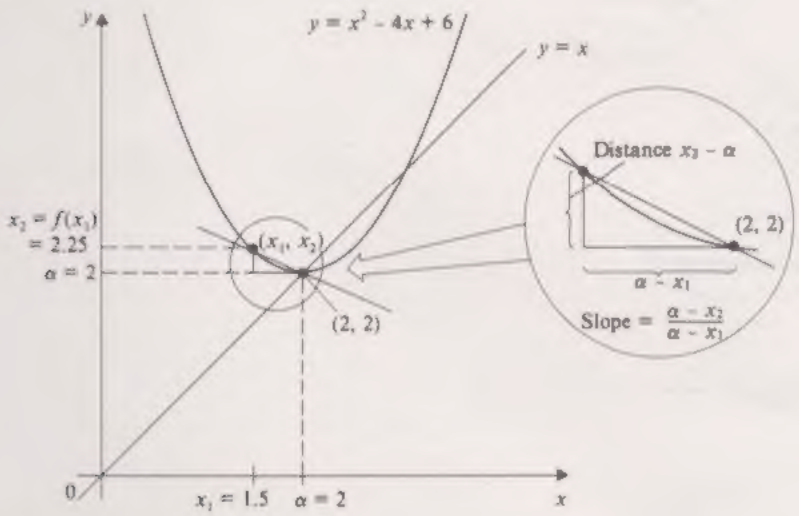
$$|\alpha - x_2| < |\alpha - x_1|.$$

Similarly, x_3 being better than x_2 as an approximation to α corresponds to

$$|\alpha - x_3| < |\alpha - x_2|,$$

which is true in this case since $|-0.0625| < |-0.25|$.

We can interpret these ideas geometrically.



The slope of the line joining (x_1, x_2) $(=(1.5, 2.25))$ to (α, α) $(=(2,2))$ is

$$\frac{\alpha - x_2}{\alpha - x_1} \quad \text{(which is } \textit{negative} \text{ in this case).}$$

Such a line joining two points on a curve is called a *chord*, and has slope $\dfrac{\text{difference in } y\text{-values}}{\text{corres. difference in } x\text{-values}}$.

We can rearrange

$$|\alpha - x_2| < |\alpha - x_1|$$

as follows:

$$|\alpha - x_1| - |\alpha - x_2| > 0;$$

$A < B$ means $B - A$ is positive.

and since $|\alpha - x_1| - |\alpha - x_2|$ and $|\alpha - x_1|$ are both positive, their quotient,

$$\frac{|\alpha - x_1| - |\alpha - x_2|}{|\alpha - x_1|},$$

is positive. Thus

$$1 - \frac{|\alpha - x_2|}{|\alpha - x_1|} > 0,$$

i.e.

$$\frac{|\alpha - x_2|}{|\alpha - x_1|} < 1.$$

Now $\frac{|\alpha - x_2|}{|\alpha - x_1|}$ is just the modulus of the slope, $\frac{\alpha - x_2}{\alpha - x_1}$, so we can interpret x_2 being better than x_1 as an approximation to α as:

> the modulus of the slope of the chord joining (x_1, x_2) to (α, α) is less than 1.

Exactly the same argument shows that x_3 being better than x_2 as an approximation to α corresponds to the modulus of the slope of the chord joining (x_2, x_3) to (α, α) being less than 1. In general, if x_{n+1} is better than x_n as an approximation to α then

$$|\text{slope of chord joining } (x_n, x_{n+1}) \text{ to } (\alpha, \alpha)| < 1.$$

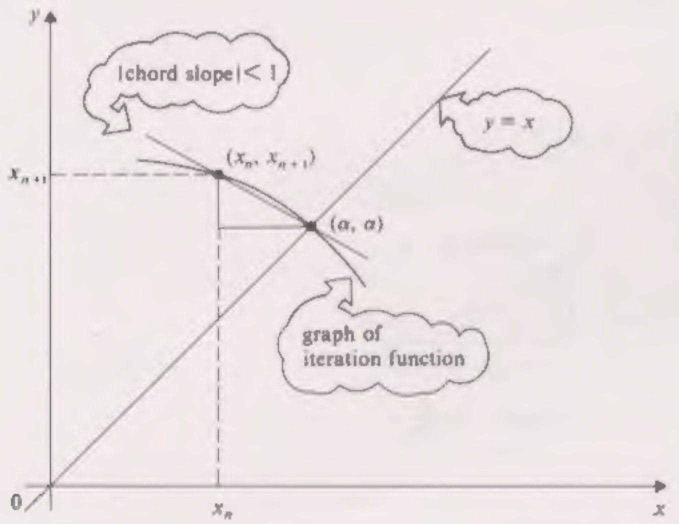
Although we have used the particular example

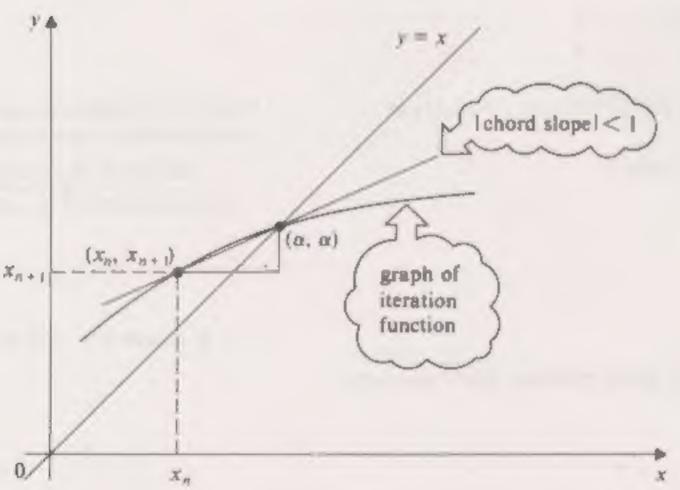
$$x_{n+1} = f(x_n) \quad \text{with} \quad f(x) = x^2 - 4x + 6,$$

the same argument holds quite generally: if x_{n+1} is closer than x_n to α, then

$$\left|\frac{\alpha - x_{n+1}}{\alpha - x_n}\right| < 1,$$

i.e. the corresponding chord has slope with modulus less than 1. Conversely, if we were able to calculate the slope of the chord joining (x_n, x_{n+1}) to (α, α) directly, and found that its modulus was less than 1, then we would know that x_{n+1} was better than x_n as an approximation to α.



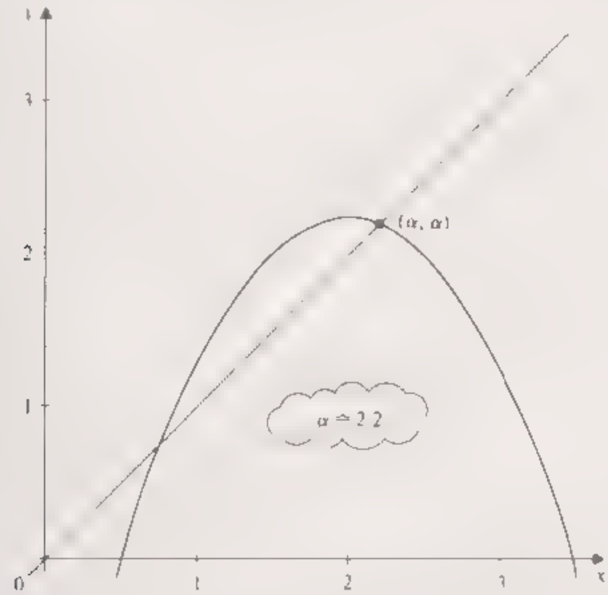


Let us now have a look at the example from Problem 4.1.3 in the light of the discussion above. It appeared, in Solution 4.1.3, that the iteration rule

$$x_{n+1} = -x_n^2 + 4x_n - \tfrac{7}{4}$$

produced a sequence of approximations converging to the larger solution of

$x = -x^2 + 4x - \frac{7}{4}$, provided that x_1 was chosen sufficiently close to this solution.



It is clear that there is a solution of $x = -x^2 + 4x - \frac{7}{4}$ in the interval $[2, 2.5]$; we shall call this solution α. We shall now show that, if we take a starting value x_1 *anywhere* in $[2, 2.5]$, then x_2 is better than x_1 as an approximation to α, i.e. we shall show that

You can justify this statement by showing that $(-x^2 + 4x - \frac{7}{4}) - x$ is positive at one end of the interval and negative at the other end.

$$\left|\frac{\alpha - x_2}{\alpha - x_1}\right| < 1.$$

We calculate x_2 from x_1 by

$$x_2 = -x_1^2 + 4x_1 - \tfrac{7}{4},$$

and α is a solution of $x = -x^2 + 4x - \frac{7}{4}$, so

$$\alpha = -\alpha^2 + 4\alpha - \tfrac{7}{4}.$$

Thus, the modulus of the slope of the chord joining (x_1, x_2) to (α, α) is

$$\left|\frac{\alpha - x_2}{\alpha - x_1}\right| = \left|\frac{(-\alpha^2 + 4\alpha - \frac{7}{4}) - (-x_1^2 + 4x_1 - \frac{7}{4})}{\alpha - x_1}\right|$$

$$= \left|\frac{-\alpha^2 + x_1^2 + 4\alpha - 4x_1}{\alpha - x_1}\right|$$

We can simplify this expression by using the fact that $\alpha^2 - x_1^2 = (\alpha - x_1)(\alpha + x_1)$. We then have

$$\left|\frac{-\alpha^2 + x_1^2 + 4\alpha - 4x_1}{\alpha - x_1}\right| = \left|\frac{-(\alpha^2 - x_1^2) + 4(\alpha - x_1)}{\alpha - x_1}\right|$$

$$= \left|\frac{-(\alpha - x_1)(\alpha + x_1) + 4(\alpha - x_1)}{\alpha - x_1}\right|$$

$$= |-(\alpha + x_1) + 4|.$$

In order to have a chord, we need (x_1, x_2) and (α, α) to be distinct points. Thus $\alpha \neq x_1$, and we may divide by the non-zero term $\alpha - x_1$.

We know that α and x_1 are in the interval $[2, 2.5]$ and that α is neither 2 nor 2.5. Hence $\alpha + x_1$ lies somewhere strictly between 4 and 5. Thus $4 - (\alpha + x_1)$ lies strictly between 0 and -1. So, certainly,

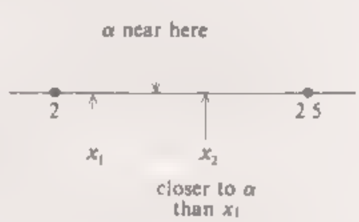
$$|-(\alpha + x_1) + 4| < 1,$$

i.e. the modulus of the slope of the chord is less than 1, justifying our assertion that x_2 is better than x_1 as an approximation to α.

We now know quite a lot about the iteration using the formula

$$x_{n+1} = -x_n^2 + 4x_n - \tfrac{7}{4}.$$

If we start with x_1 reasonably close to the solution α in $[2, 2.5]$, then x_2 will be closer than x_1 to α and, almost certainly, also lie in $[2, 2.5]$ because α ($\simeq 2.2$) is near the middle of $[2, 2.5]$. Provided that x_2 is in $[2, 2.5]$, the same type of chord-slope argument as above will show that x_3 is closer than x_2 to α, and so on.



In this case, not only do the distances between $x_1, x_2, \ldots$ and α, i.e.

$$|\alpha - x_1|, |\alpha - x_2|, \ldots,$$

decrease in size, but they actually tend to zero, so that the sequence of approximations

$$x_1, x_2, \ldots$$

converges to α. (For general sequences, we must be careful of drawing such conclusions, as the following example indicates: the distances between the terms of the sequence

$$2\tfrac{1}{2}, 2\tfrac{1}{3}, 2\tfrac{1}{4}, \ldots, 2 + \frac{1}{n}, \ldots$$

and 1 *do* decrease in size but *do not* tend to zero.)

Let us review what we have done in the more general context of finding out when a formula iteration converges. The modulus of the slope of the chord joining the point

$$(x_n, x_{n+1})$$

to the "solution point"

$$(\alpha, \alpha)$$

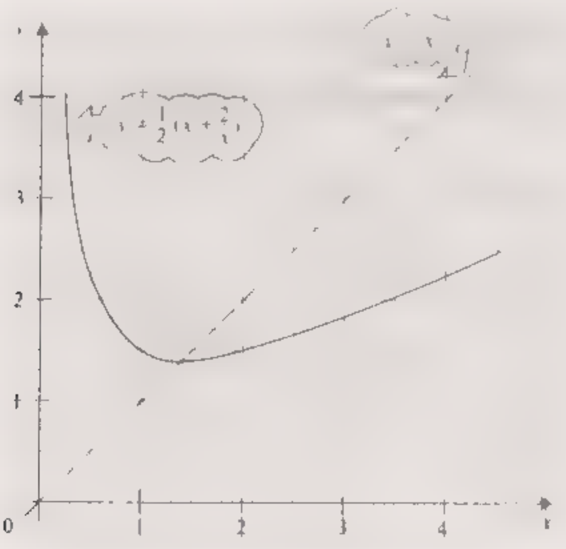
determines whether or not x_{n+1} is better than x_n as an approximation to α. It seems very likely that if all chords passing through (α, α) and points reasonably close to (α, α) have slope with modulus less than 1, *and* we start the iteration reasonably close to α, then the iteration will converge.

If "reasonably close" is made a little more precise, this statement is, indeed, true, but a full justification is well beyond the scope of a foundation course.

Thus, to test a given formula iteration for likely success, we should have to calculate slopes of chords. In anything more than very simple cases, this is a daunting prospect. In Section 4.3 of this unit we shall develop a more easily applied test; meanwhile, we close this section with a problem which reviews the basic ideas about chord slopes.

Problem 4.1.4

(i) The diagram below shows part of the graphs of $y = x$ and $y = \frac{1}{2}\left(x + \frac{2}{x}\right)$.



(a) From the graphs, write down, to one decimal place, an approximate value of the positive solution, α, of the equation

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

(b) Use the diagram to predict the behaviour of the iteration rule

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

with starting value $x_1 = 1$.

(ii) (a) Write down an expression for the slope of the chord joining (x_1, x_2) and (α, α), and use your approximate value for α to estimate the modulus of this chord slope

(b) Repeat part (ii)(a) for the chord joining (x_2, x_3) and (α, α).

(c) Find an expression (in terms of x_n and α) for the slope of the chord joining (x_n, x_{n+1}) and (α, α). By using the fact that α satisfies

$$\alpha = \frac{1}{2}\left(\alpha + \frac{2}{\alpha}\right),$$

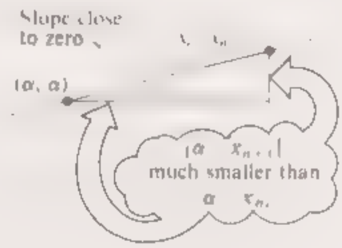
show that the modulus of this chord slope is

$$\left|\frac{1}{2} - \frac{1}{x_n \alpha}\right|,$$

and estimate this modulus when x_n is close to α.

Solution 4.1.4 illustrates two things. Firstly, the involved algebra required to investigate even a fairly simple formula iteration. Secondly, that chord slopes close to zero near a "solution point" (α, α) are a very good thing. The modulus of the chord slope in part (ii)(c) is

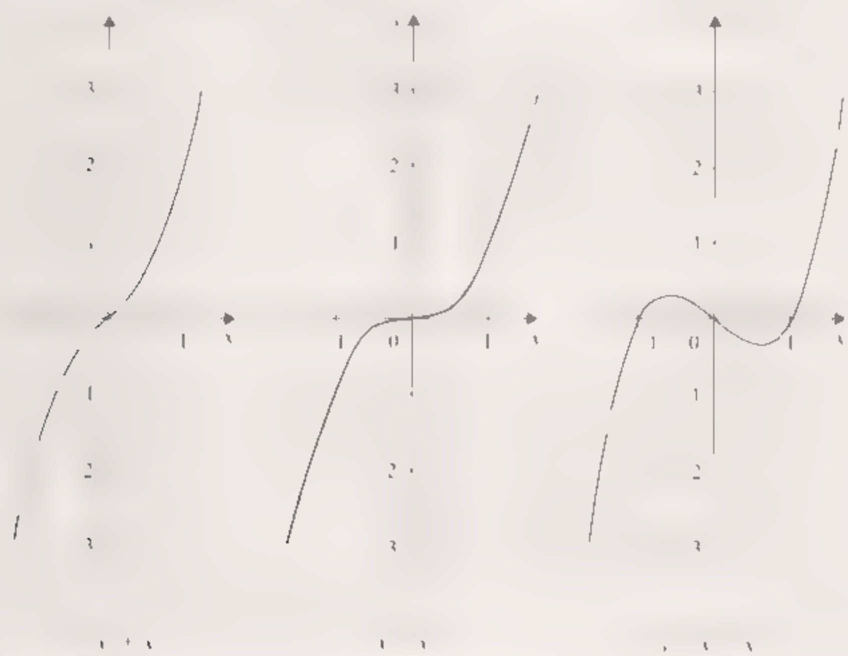
$$\left|\frac{\alpha - x_{n+1}}{\alpha - x_n}\right| = \left|\frac{1}{2} - \frac{1}{x_n \alpha}\right|,$$



and if this is very close to zero then the numerator, $|\alpha - x_{n+1}|$, must be much closer to zero than is $|\alpha - x_n|$, i.e. x_{n+1} is much better than x_n as an approximation to α

4.2 TANGENTS

In this section we take up one of the problems posed in *Unit 1* of this block — that of giving an algebraic definition of what we mean by a tangent to the graph of a function. In that unit we stated that our three basic cubic curves,



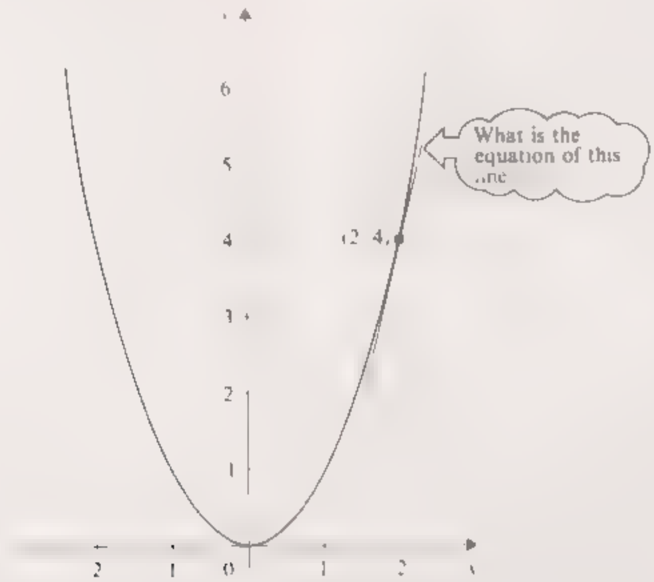
had, respectively, no horizontal tangents, one horizontal tangent (where $x = 0$) and two horizontal tangents $\left(\text{where } x = \frac{-1}{\sqrt{3}} \text{ and } x = \frac{1}{\sqrt{3}}\right)$. The methods we shall now discuss will enable us to justify these statements.

This is the same as we gave in *Unit 1* because

$$\frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Rather than start with cubics, we shall explore the basic ideas by looking at a simpler graph — that of $y = x^2$. To get into the problem, let us look at a specific question:

What is the equation of the tangent to the graph of $y = x^2$ at the point (2,4)?



What do we mean by this tangent? First of all, it is a line *through* the point (2,4) on the curve. Of all such lines, we want to select the one which just touches the curve at (2, 4). We can write down the equation of a line through (2, 4) once we know what its slope is. Let us assume, then, that the slope is m; then the equation of the line must be of the form

$$y = mx + c$$

for some value of c. However, the line is to pass through (2, 4) so

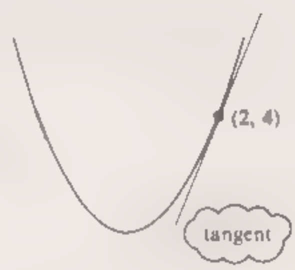
$$4 = m \times 2 + c,$$

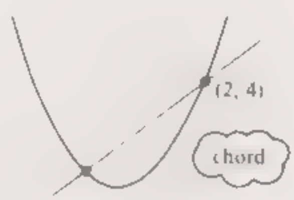
i.e.

$$c = 4 - 2m.$$

Hence the equation of the line through (2, 4) with slope m is

$$y = mx + (4 - 2m).$$





The tangent intersects the curve in just one point, whereas other lines through (2, 4) cut the curve in two points. To find out where

$$y = mx + (4 - 2m)$$

intersects

$$y = x^2,$$

we solve the equation

$$x^2 = mx + (4 - 2m).$$

Rearranging this, we obtain

$$x^2 - mx + 2m - 4 = 0.$$

We already know that one point of intersection occurs when $x = 2$, so this quadratic equation can be factorized as

$$(x - 2)(x \ldots) = 0.$$

To get the constant term, $2m - 4$, correct, the factors must be

$$(x - 2)(x + 2 - m) = 0.$$

You can check this by multiplying out the brackets.

The two solutions are

$$x = 2 \quad \text{and} \quad x = m - 2.$$

If the line is to be a tangent, there must be only one point of intersection, and this will happen when $x = 2$ and $x = m - 2$ are the same. This requires

$$2 = m - 2,$$

i.e.

$$m = 4.$$

Hence the tangent to $y = x^2$ at the point (2, 4) has slope 4 and equation

$$y = 4x + (4 - 2 \times 4),$$

i.e.

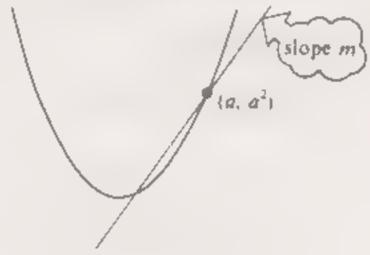
$$y = 4x - 4.$$

Problem 4.2.1

Use the above technique to find the slope of the tangent to $y = x^2$ at the point $(3,9)$. Hence write down the equation of this tangent.

We can carry out this same procedure starting with an arbitrary point (a, a^2) on the graph of $y = x^2$. By doing so, we shall obtain a general formula for the slope of the tangent to $y = x^2$ at the point (a, a^2). Particular results can then be obtained by simply substituting in the appropriate value of a.

Once we know the slope, it is easy to find the equation of the tangent.



A line with slope m has an equation of the form

$$y = mx + c.$$

The point (a, a^2) must be on this line, so

$$a^2 = ma + c,$$

i.e.

$$c = a^2 - ma.$$

Thus the line through (a, a^2) with slope m has the equation

$$y = mx + (a^2 - ma).$$

We find the points where this line intersects $y = x^2$ by solving

$$x^2 = mx + (a^2 - ma),$$

i.e.

$$x^2 - mx + ma - a^2 = 0.$$

We know that $x = a$ is one solution, giving the corresponding factor $x - a$:

$$(x - a)(x \ldots) = 0.$$

To get the constant term, $ma - a^2$, correct, the second factor must be $(x + a - m)$, giving:

$$(x - a)(x + a - m) = 0,$$

with solutions

$$x = a, \quad x = m - a.$$

You can check this by multiplying out the brackets.

If the line is to be a tangent, these two values must be the same, which means that

$$a = m - a.$$

i.e.

$$m = 2a$$

Thus the tangent to $y = x^2$ at the point (a, a^2) has slope $2a$.

Problem 4.2.2

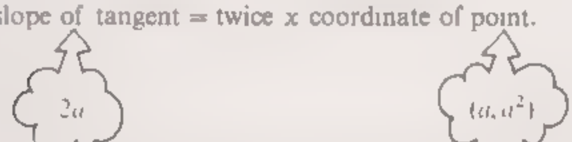
(i) Use the above result to write down the slope of the tangent to $y = x^2$ at (a) $(-1, 1)$; (b) $(\frac{1}{2}, \frac{1}{4})$.

(ii) Find the equations of the tangents in part (i).

The result can be summed up as:

for $y = x^2$

slope of tangent = twice x coordinate of point.



So, to each x-value a in $\mathbb{R}$, there is associated the slope, $2a$, of the tangent at the corresponding point (a, a^2) on the curve. In other words, we have defined a "slope-function" with rule

$$a \longmapsto \text{slope of tangent at } x = a$$

i.e. $\quad a \longmapsto 2a.$

We may perfectly well denote this rule by $t \longmapsto 2t$ or $u \longmapsto 2u$ or $x \longmapsto 2x$, etc.

We started with the graph of the function

$$\mathbb{R} \longrightarrow \mathbb{R},$$
$$x \longmapsto x^2,$$

and we now have the "slope-function"

$$\mathbb{R} \longrightarrow \mathbb{R},$$
$$x \longmapsto 2x.$$

The standard name for the slope-function is the *derived function*. Thus the derived function of $x \longmapsto x^2$ $(x \in \mathbb{R})$ is $x \longmapsto 2x$ $(x \in \mathbb{R})$.

There is also a standard notation for derived functions: the derived function of a function labelled f is denoted by f'. Thus, in our example above, if

Read "f-dashed" or "f-prime".

$$f : x \longmapsto x^2 \qquad (x \in \mathbb{R}),$$

then the derived function of f is

$$f' : x \longmapsto 2x \qquad (x \in \mathbb{R}).$$

Similarly, the derived function of a function g would be denoted by g', and so on.

Problem 4.2.3

Find the derived function of $g : x \longmapsto 4x - x^2$ $(x \in \mathbb{R})$ by carrying out the following steps.

(i) Find the equation of the line through $(a, 4a - a^2)$ with slope m.

(ii) Find the equation satisfied by the x coordinates of the points of intersection of this line and the graph of $y = 4x - x^2$.

(iii) Solve the equation in part (ii) and hence find, in terms of a, the value of m for which the line and the curve intersect in just one point.

(iv) Complete: (a) the slope of the tangent to $y = 4x - x^2$ at $x = a$ is $\boxed{}$;

(b) the derived function of g is $g' : x \longmapsto \boxed{}$ $(x \in \mathbb{R})$.

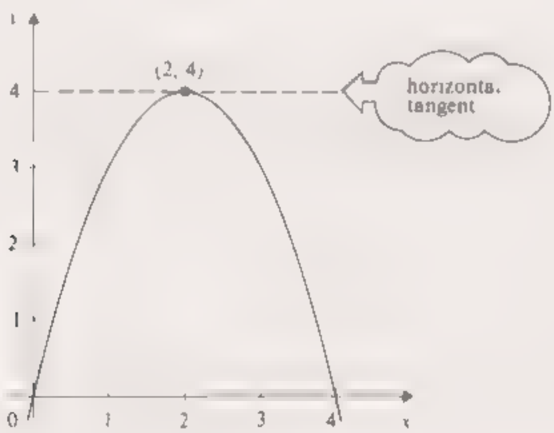
From Solution 4.2.3, we can observe that the tangent to $y = 4x - x^2$ has slope zero when $x = 2$ because then the slope of the tangent is

$$g'(2) = 4 - 2 \times 2 = 0.$$

Lines with zero slope are "horizontal" (i.e. parallel to the x-axis). The result that $y = 4 - x^2$ has a horizontal tangent at $(2, 4)$ corresponds to what we should expect from the completed square form,

$$4x - x^2 = -(x - 2)^2 + 4,$$

which indicates that the graph is as shown below.



It seems reasonable, therefore, to justify the assertions about the three basic cubic graphs by obtaining the derived functions of

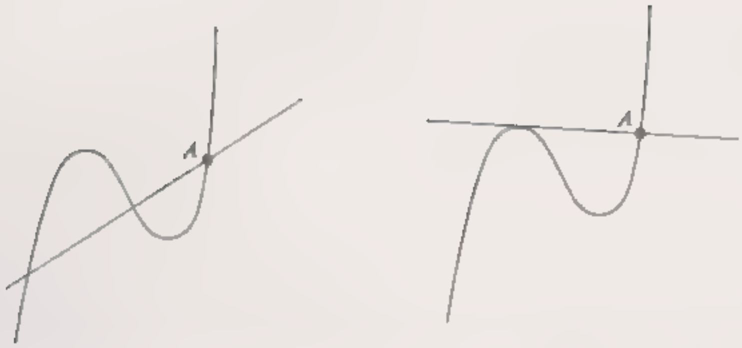
$$x \longmapsto x^3 + x \qquad (x \in \mathbb{R})$$

$$x \longmapsto x^3 \qquad (x \in \mathbb{R})$$

and $\quad x \longmapsto x^3 - x \qquad (x \in \mathbb{R})$

and then finding out where, if ever, the tangents to the graphs have slope zero. Unfortunately, the method we have used for finding derived functions of quadratics will not generalize to cubics.

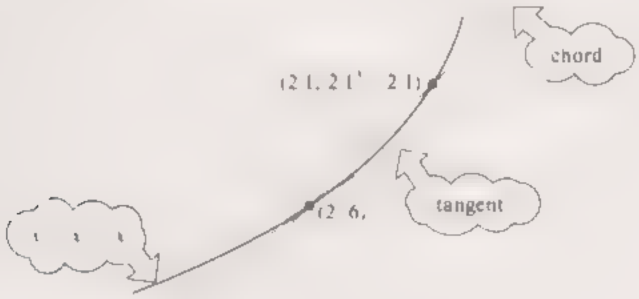
If we are looking for the tangent at A, many lines through A intersect the curve in three points. Even if we can find a line intersecting the curve at A and only one other point, it need not be the tangent at A.



We shall use a method based on approximating the slope of the tangent by slopes of chords. To illustrate the idea we shall look at a very specific example:

What is the slope of the tangent to $y = x^3 - x$ at the point $(2, 6)$?

To begin with, we shall approximate the tangent at $(2, 6)$ by the chord joining $(2, 6)$ to $(2.1, (2.1)^3 - (2.1))$. A sketch of the situation is shown below.



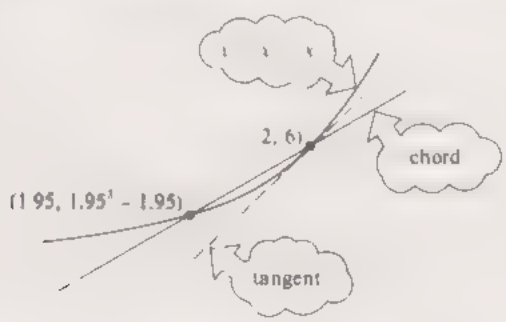
The slope of the chord is

$$\frac{\text{difference in } ys}{\text{corresponding difference in } xs} = \frac{(2.1^3 - 2.1) - 6}{2.1 - 2}$$

$$= \frac{1.161}{0.1}$$

$$= 11.61.$$

We can obtain another estimate for the slope of the tangent by taking another point, closer to $(2, 6)$.



This time we have taken an x-value slightly less than 2 for the other point on the graph.

This time the slope of the chord is

$$\frac{\text{difference in } ys}{\text{corresponding difference in } xs} = \frac{(1.95^3 - 1.95) - 6}{1.95 - 2}$$

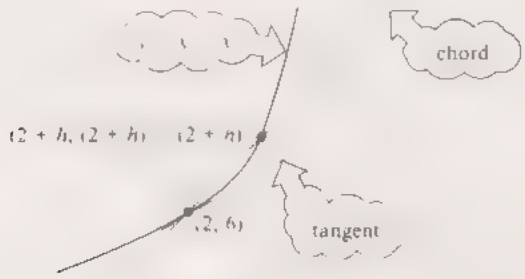
$$= \frac{-0.535125}{0.05}$$

$$= 10.7025.$$

The diagrams suggest that the slope of the tangent lies somewhere between the two approximations 10.7025 and 11.61. It also seems reasonable that a chord joining $(2, 6)$ to a point *very close to* $(2, 6)$ would be a *good* approximation to the

tangent. Rather than try more numerical examples, it is now sensible to generalize and consider a point near $(2, 6)$ with coordinates

$$(2 + h, (2 + h)^3 - (2 + h)).$$



Our two numerical examples correspond to $h = 0.1$ and $h = -0.05$. The diagram illustrates a positive value of h but, as we have seen, h could be negative. We cannot take $h = 0$ because we need two distinct points to obtain a chord.

The slope of the chord is

$$\frac{\text{difference in } ys}{\text{corresponding difference in } xs} = \frac{((2 + h)^3 - (2 + h)) - 6}{(2 + h) - 2}$$

$$= \frac{8 + 12h + 6h^2 + h^3 - 2 - h - 6}{h}$$

$$= \frac{11h + 6h^2 + h^3}{h}$$

$$= 11 + 6h + h^2.$$

This algebra has revealed something which might not have been apparent even if we had done many more numerical examples: the slope of the chord joining

$$(2, 6) \quad \text{to} \quad (2 + h, (2 + h)^3 - (2 + h))$$

is

$$11 + 6h + h^2,$$

and the nearer to zero h is, the closer this slope is to 11. By taking h sufficiently close to zero, the slope of the chord may be made as close to 11 as we please. Thus we conclude that the slope of the tangent to $y = x^3 - x$ at $(2, 6)$ is 11.

We cannot take $h = 0$ because one step in obtaining the chord slope was to divide by h.

The advantage of this technique of approximating tangents by chords is that it will generalize to functions other than cubics; an advantage that we pursue later.

We can now use the above technique to find the derived function of $f: x \longmapsto x^3 - x$ by considering $(a, a^3 - a)$ on the graph $y = x^3 - x$, rather than the special case $(2, 6)$. We shall set out the calculation in a systematic form that will be useful later.

Take point near $(a, a^3 - a)$ with x coordinate $a + h$: $\quad (a + h, (a + h)^3 - (a + h)).$

Find slope of chord joining points

as $\dfrac{\text{difference in } ys}{\text{corresponding difference in } xs}$: $\quad \dfrac{((a + h)^3 - (a + h)) - (a^3 - a)}{(a + h) - a}$

$$= \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a - h - a^3 + a}{h}$$

(using the Binomial Theorem)

$$= \frac{3a^2h + 3ah^2 + h^3 - h}{h}$$

$$= 3a^2 + 3ah + h^2 - 1$$

$$= (3a^2 - 1) + 3ah + h^2.$$

Investigate the value of the slope as h is made close to zero: only the term $3a^2 - 1$ is unchanged; $3ah$ and h^2 will become close to zero as h is made close to zero.

We conclude that the slope of the tangent to the curve

$$y = x^3 - x$$

at the point

$$(a, a^3 - a)$$

is

$$3a^2 - 1.$$

Thus, with the function

$$f: x \longmapsto x^3 - x \qquad (x \in \mathbb{R})$$

we have associated the slope function $a \longmapsto 3a^2 - 1$ $(a \in \mathbb{R})$, i.e. we have found the derived function

$$f': x \longmapsto 3x^2 - 1 \qquad (x \in \mathbb{R}).$$

We can now justify the assertion that the graph of $y = x^3 - x$ has two horizontal tangents. The tangent at $(x, x^3 - x)$ on the graph has slope $3x^2 - 1$. This slope is zero when

$$3x^2 - 1 = 0,$$

i.e. when $\quad x^2 = \frac{1}{3}$

or $\quad x = \pm\sqrt{\tfrac{1}{3}}$

$$= \pm \frac{1}{\sqrt{3}}$$

Thus the tangent to $y = x^3 - x$ will be horizontal at the points

$$\left(-\frac{1}{\sqrt{3}}, \left(-\frac{1}{\sqrt{3}}\right)^3 - \left(-\frac{1}{\sqrt{3}}\right)\right) \text{ and } \left(\frac{1}{\sqrt{3}}, \left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{1}{\sqrt{3}}\right)\right),$$

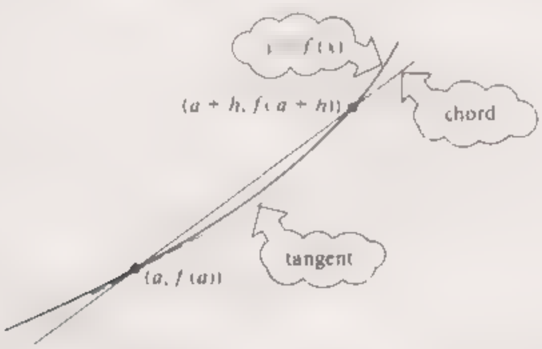
i.e. $$\left(-\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}}\right) \text{ and } \left(\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}}\right).$$

$$\left(-\frac{1}{\sqrt{3}}\right)^3 - \left(-\frac{1}{\sqrt{3}}\right)$$
$$= -\frac{1}{(\sqrt{3})^3} + \frac{1}{\sqrt{3}}$$
$$= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}}$$
$$= \frac{2}{3} \times \frac{1}{\sqrt{3}} \text{ etc}$$

These results correspond to the assertions we made, in *Unit 1* of this block, about the location of the peak and trough of the graph of $y = x^3 - x$.

The above technique for finding slopes of tangents, and hence derived functions, is so important that we summarize it now in its general form.

To find the slope of the tangent to the graph of $y = f(x)$ at $(a, f(a))$.



1. Write down the coordinates of the point on the graph with x coordinate $a + h$, i.e. $(a + h, f(a + h))$.
2. Write down the slope of the chord joining $(a, f(a))$ to $(a + h, f(a + h))$. This slope is

$$\frac{f(a+h) - f(a)}{h}.$$

difference in ys

difference in xs, $(a + h) - a$

3. Investigate what happens to this chord slope as h is made close to zero.

The process in Step 3 is referred to as "letting h tend to zero". Thus in Step 3 the question is whether

$$\frac{f(a+h)-f(a)}{h}$$

tends to some value as h tends to zero. If so, then the value obtained is the slope of the tangent

Problem 4.2.4

Find the slope of the tangent at (a, a^3) on the graph of $y = x^3$, and hence find the derived function of $g: x \longmapsto x^3$ $(x \in \mathbb{R})$, by completing the following

(i) The coordinates of $(a+h, g(a+h))$ are $(a+h, (a+h)^3)$, i.e.

$$(a+h, \boxed{}).$$

[Hint: You may find the Binomial Theorem useful]

(ii) The slope of the chord joining $(a, g(a))$ to $(a+h, g(a+h))$ is

$$\frac{g(a+h)-g(a)}{h} = \frac{\boxed{} - a^3}{h}$$

(iii) As h tends to zero, this chord slope tends to $\boxed{}$.

(iv) The slope of the tangent at (a, a^3) on the graph of $y = x^3$ is $\boxed{}$.

(v) The derived function, g', of g is

$$g': x \longmapsto \boxed{} \quad (x \in \mathbb{R}).$$

Problem 4.2.5

Use Solution 4.2.4 to find the coordinates of the point(s), if any, where $y = x^3$ has a horizontal tangent.

Problem 4.2.6

Repeat Problems 4.2.4 and 4.2.5 for the graph $y = x^3 + x$ and the function $k: x \longmapsto x^3 + x$ $(x \in \mathbb{R})$.

We have now justified the horizontal tangent properties of all three basic cubic graphs by finding the derived functions of the corresponding functions. The technique for finding derived functions, by looking at the value of

$$\frac{f(a+h)-f(a)}{h}$$

as h tends to zero, is the foundation of much of the work that we shall do in Block III

In the next section we shall look at the link between formula iteration and derived functions. Later in this unit we shall extend the list of derived functions that we have obtained so far.

Rule for function	Rule for derived function
$x \longmapsto x^2$	$x \longmapsto 2x$
$x \longmapsto x^3$	$x \longmapsto 3x^2$
$x \longmapsto 4x - x^2$	$x \longmapsto 4 - 2x$
$x \longmapsto x^3 - x$	$x \longmapsto 3x^2 - 1$
$x \longmapsto x^3 + x$	$x \longmapsto 3x^2 + 1$

4.3 ITERATION AND CONVERGENCE

In this section we make use of the idea of derived function from Section 4.2 to obtain a test for convergence of an iteration formula with a given starting value. The pre-programme work includes a number of problems whose results are referred to in the TV programme.

Pre-programme Work

Problem 4.3.1

(i) Show that

$$x = x^3 - \frac{11}{16}x + \frac{3}{4}$$

is a rearrangement of

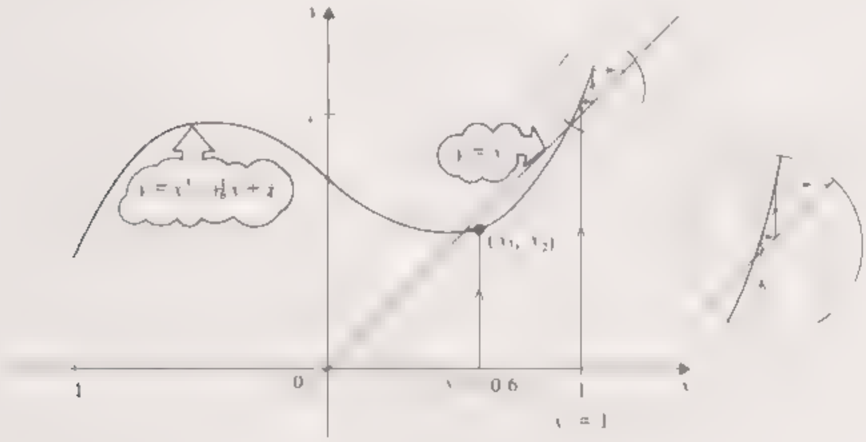
$$x^3 - \frac{27}{16}x + \frac{3}{4} = 0$$

(ii) Find $x_2, x_3, \ldots, x_8$ for the iteration formula

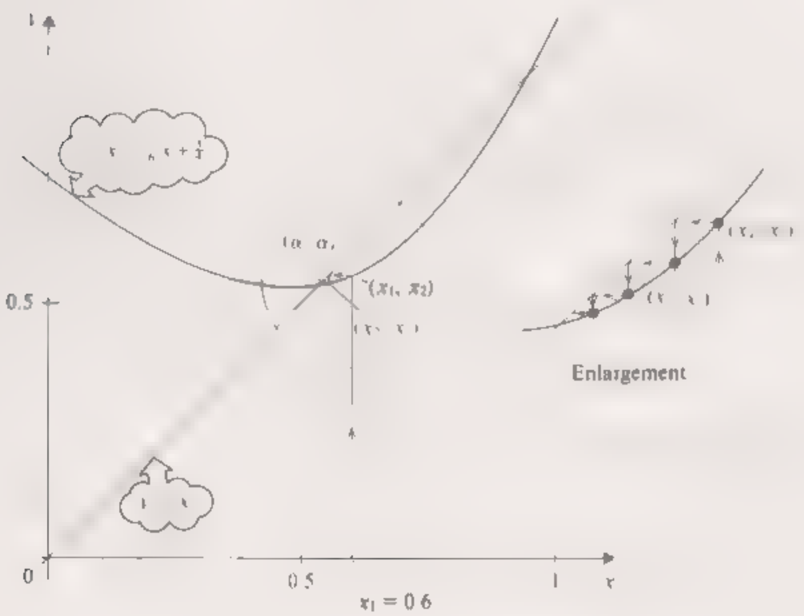
$$x_{n+1} = x_n^3 - \frac{11}{16}x_n + \frac{3}{4}$$

for (a) $x_1 = 0.6$, (b) $x_1 = 1$.

The results of Solution 4.3.1 can be illustrated by the corresponding "staircase and cobweb" diagrams.



Looking at the case $x_1 = 0.6$ in more detail



shows how the formula iteration produces a sequence of points

$$(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_{n+1})$$

on the graph of $y = f(x)$.

The sequence of points approaches (α, α), where α is a solution of the equation

$$x = f(x),$$

i.e.

$$x = x^3 - \frac{11}{16}x + \frac{3}{4},$$

in other words,

$$x^3 - \frac{27}{16}x + \frac{3}{4} = 0.$$

Problem 4.3.2

(i) Find the slope of the tangent to the graph of

$$y = f(x), \quad f(x) = x^3 - \frac{11}{16}x + \frac{3}{4}$$

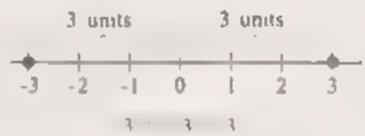
at the point $(a, f(a))$ by the method outlined in Section 4.2 of this unit. (That is, calculate $\dfrac{f(a+h) - f(a)}{h}$, $h \neq 0$, and find out what happens to this expression as h is made closer and closer to zero.)

(ii) Write down the rule for the derived function f' of

$$f(x) = x^3 - \frac{11}{16}x + \frac{3}{4} \qquad (x \in \mathbb{R}).$$

The TV programme involves handling inequalities. We require one technique beyond those in *Unit 2*, for solving inequalities involving the modulus function.

To begin with, you may recall that one way of thinking of the modulus of x, $|x|$, is as the distance of x from the origin on a number line; thus 3 and -3 are both 3 units from the origin and this is reflected in the fact that $|3| = |-3| = 3$.



Problem 4.3.3

(i) Find the solution set of the inequality $|x| < 4$, giving your answer in interval notation.

(ii) Write down the solution to part (i) using inequalities (but not the modulus sign).

Problem 4.3.4

Find the solution set of $|x - 1| < 3$. (Hint: When is the distance of $x - 1$ from the origin less than 3?)

We shall apply the technique of replacing

$$|c| < d \quad (d \text{ a positive constant})$$

by $\quad -d < c < d$

or, equivalently,

$$-d < c \quad \text{and} \quad c < d$$

to inequalities of the form

$$|\text{expression involving } x| < 1.$$

(We shall be particularly concerned with the case where the expression involving x is $f'(x)$ for some function f.)

Problem 4.3.5

The iteration function in Problem 4.3.1 was

$$f(x) = x^3 - \frac{11}{16}x + \frac{3}{4},$$

and in Problem 4.3.2 you found its derived function:

$$f'(x) = 3x^2 - \frac{11}{16}.$$

Find the values of x for which

$$|f'(x)| < 1,$$

by solving

$$\left|3x^2 - \frac{11}{16}\right| < 1,$$

expressing your answer in interval notation.

(You may find it helpful to use the "difference of two squares" factorization $x^2 - a^2 = (x - a)(x + a)$.)

Review of Programme

The programme opened by reviewing the behaviour of

$$x_{n+1} = x_n^3 - \frac{11}{16}x_n + \frac{3}{4}$$

This corresponds to your investigation in Problem 4.3.1.

for starting values of 0.6 (which gave a convergent sequence) and 1 (which gave a divergent sequence). The results were illustrated by cobweb staircase diagrams. The main aim of the programme was to find a test which would *guarantee* the convergence of a formula iteration for a given starting value.

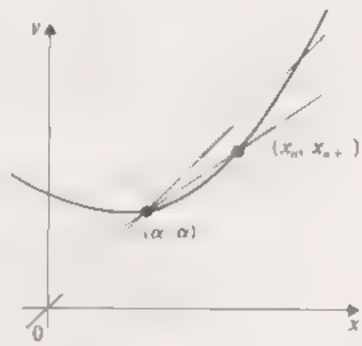
x_n $f(x_n)$
starting value x
converges if
?

The first attempt was to require x_{n+1} to be closer than x_n to α, for every $n = 1, 2, \ldots$, i.e.

$$|x_{n+1} - \alpha| < |x_n - \alpha|.$$

As in Section 4.1, this was rewritten as

$$\left|\frac{x_{n+1} - \alpha}{x_n - \alpha}\right| < 1,$$



and interpreted as a statement about the modulus of the slope of the chord joining (x_n, x_{n+1}) to (α, α).

Thus a first attempt at a test for convergence was to check that the formula and starting values lead to

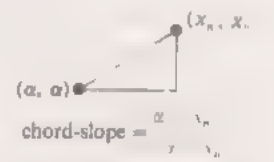
$$|\text{slope of chord from } (x_n, x_{n+1}) \text{ to } (\alpha, \alpha)| < 1,$$

for all values of $n = 1, 2, \ldots$.

As we mentioned briefly in Section 4.1, this test is impractical. The value of α is not known, and so checking

$$|\text{chord-slope}| < 1$$

will be difficult

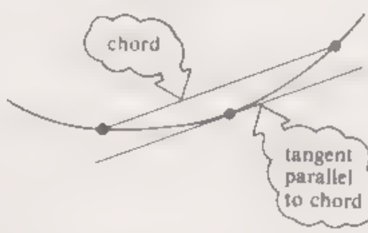


To overcome this difficulty, attention was transferred to tangent-slopes. For our curve, any chord of the graph has a parallel tangent attached at a point between the ends of the chord. This step was of great importance because techniques for finding slopes of tangents are available. If it is known that

$$|\text{tangent-slopes}| < 1$$

for some interval

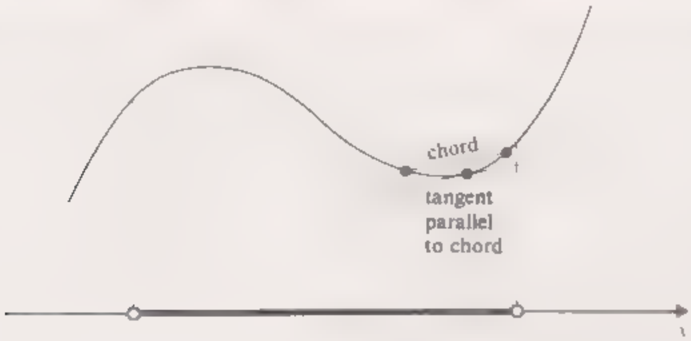
x_n $f(x_n)$
starting value x_1
converges if
|chord-slope|<1



|tangent-slopes|<1

then the slopes of chords with end-points in the interval must also have modulus less than 1.

The key steps in the argument were as follows. The chord, with end-points in the interval,



is parallel to a tangent attached at a point also in the interval, so

$$|\text{tangent-slope}| < 1$$

and hence

$$|\text{chord-slope}| < 1.$$

We then considered the test

$$|\text{tangent-slope}| < 1$$

for the formula

$$x_{n+1} = x_n^3 - \frac{11}{16}x_n + \frac{3}{4}$$

i.e. $$f(x) = x^3 - \frac{11}{16}x + \frac{3}{4}$$

The derived function gives the slope of the tangent, and here

$$f'(x) = 3x^2 - \frac{11}{16}.$$

As in Solution 4.3.2.

The inequality

$$|f'(x)| < 1$$

was solved to find out where it was true that

$$|\text{tangent-slopes}| < 1.$$

The technique of Solution 4.3.4 was used to obtain

$$|f'(x)| < 1 \quad \text{for } x \in \left]-\tfrac{3}{4}, \tfrac{3}{4}\right[$$

The first thing checked was that

$$x = x^3 - \frac{11}{16}x + \frac{3}{4}$$

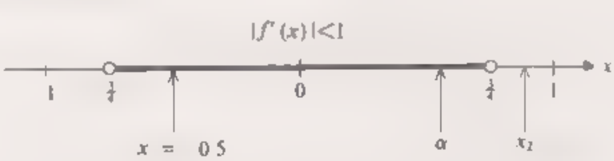
actually has a solution in $]-\frac{3}{4}, \frac{3}{4}[$. This was done by checking that the graph of $y = x^3 - \frac{11}{16}x + \frac{3}{4}$ lies above that of $y = x$ at $x = -\frac{3}{4}$ and below at $x = \frac{3}{4}$. Alternatively, you could check that $x^3 - \frac{27}{16}x + \frac{3}{4}$ changes sign on $[-\frac{3}{4}, \frac{3}{4}]$

Thus, starting with x_1 in $]-\frac{3}{4}, \frac{3}{4}[$, x_2 must be closer than x_1 to the solution α. The starting value $x_1 = 0.6$ is in $]-\frac{3}{4}, \frac{3}{4}[$, and that iteration did converge; $x_1 = 1$ is outside $]-\frac{3}{4}, \frac{3}{4}[$ and that iteration diverged.

The next starting value considered was $x_1 = -0.5$ (which is in the interval $]-\frac{3}{4}, \frac{3}{4}[$). The iteration failed to converge, making the point that the condition

$$|f'(x)| < 1$$

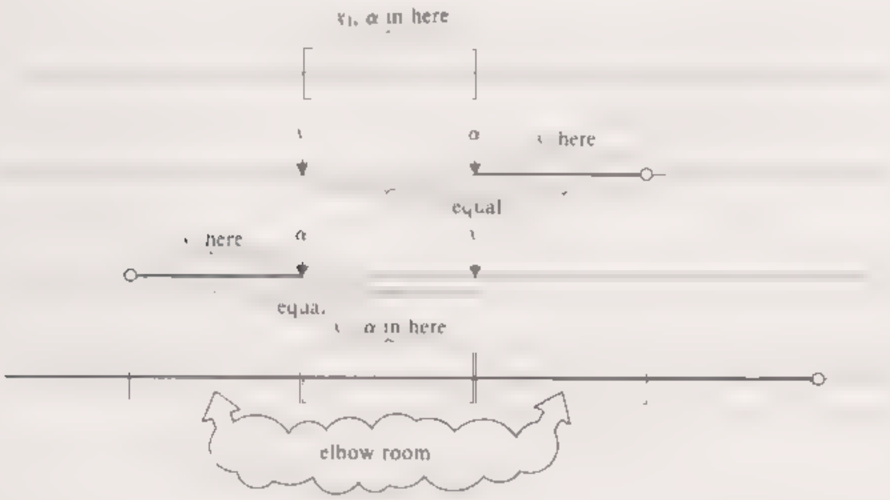
is important, but far from the whole story. The problem lay in the fact that $x_1 = -0.5$ *was* in the interval $]-\frac{3}{4}, \frac{3}{4}[$, which guaranteed that $x_2 = 0.96875$ was closer than x_1 to α ($\simeq 0.5354$) but, unfortunately, x_2 *did not* lie in the interval for which $|f'(x)| < 1$, so the condition did not guarantee that x_3 was closer than x_2 to α. In fact, x_3 was further away, and the iteration diverged



x_2 is closer to α than is x_1, but $x_2 \notin]-\frac{3}{4}, \frac{3}{4}[$

Thus there was a need for a condition to ensure that the starting value x_1 was "near enough" to α to avoid "jumping over" α and landing outside the interval where $|f'(x)| < 1$.

At this stage all we knew about α was that it was in $]-\frac{3}{4},\frac{3}{4}[$, which is too large an interval. The problem was to decide on an interval containing α which would be small enough to prevent an x_1 in the small interval giving an x_2 outside $]-\frac{3}{4},\frac{3}{4}[$. This was tackled by considering "worst possible" cases.



We found that the following condition would guarantee that the sequence never jumped out of the interval for which $|f'(x)| < 1$. Choose the starting value x_1 to be in a small interval containing α which has *at least its own length* on either side as "elbow room" within the interval for which $|f'(x)| < 1$.

Problem 4.3.6

(i) Show that $x = x^3 - \frac{11}{16}x + \frac{3}{4}$ has a solution in $[0.5, 0.6]$.

(ii) Use an "elbow room" argument to prove that the formula iteration

$$x_{n+1} = x_n^3 - \frac{11}{16}x_n + \frac{3}{4}$$

will converge for any starting value in $[0.5, 0.6]$.

The programme closed by checking the prediction given by Solution 4.3.6 for $x_1 = 0.5$.

Post-programme Work

We can summarize the test for convergence of

$$x_{n+1} = f(x_n)$$

with starting value x_1 as follows.

1. Find $f'(x)$.
2. Solve the inequality $|f'(x)| < 1$.
3. Write down a small interval in the solution set above.
4. Check that α lies in this small interval and that the interval has enough elbow room

When all the above are satisfied, *any* starting value in the small interval will always lead to convergence for polynomial functions f and other functions such as sine, cosine, logarithms and exponentials

Problem 4.3.7

Consider the formula iteration $x_{n+1} = f(x_n)$, where

$$f(x) = x^3 + \tfrac{1}{4}x + \tfrac{1}{8}.$$

(i) Given that $f'(x) = 3x^2 + \frac{1}{4}$, find the solution set of the inequality $|f'(x)| < 1$.

(ii) (a) Write the equation $x = f(x)$ in the form

expression involving $x = 0$.

(b) Show that the equation in part (a) has a solution, α, in the interval $[0.1, 0.2]$

(iii) Prove that iteration with starting value 0.1 will converge

(iv) Find the value of α to four decimal places.

Problem 4.3.8

Consider the formula iteration $x_{n+1} = f(x_n)$, where

$$f(x) = x^3 - 3x^2 + \tfrac{8}{3}x + \tfrac{1}{6}.$$

(i) Given that $f'(x) = 3x^2 - 6x + \frac{8}{3}$, find the solution set of $|f'(x)| < 1$. (Hint: $x^2 - 2x + \frac{5}{9}$ factorizes as $(x - \frac{1}{3})(x - \frac{5}{3})$.)

(ii) Write the equation $x = f(x)$ in the form expression $= 0$, and hence show that there is a solution in $[\frac{2}{3}, 1]$.

(iii) Prove that iteration with starting value 0.7 will converge.

That completes the main theme of this section, but we should like to draw your attention to some points that you may well have noticed already.

Every time the term x^3 appears in a function, $3x^2$ appears in the derived function; more than that, the terms in the derived function seem to arise from the separate terms in the function, with the exception that constants seem to vanish. In order to make use of derived functions for testing iteration, we need a systematic investigation into how derived functions are built up; finding them from first principles takes too long for regular use. To point the way, we ask you to find derived functions for two particularly simple functions.

Problem 4.3.9

(i) Find the derived function of $f: x \longmapsto 1$ $(x \in \mathbb{R})$.

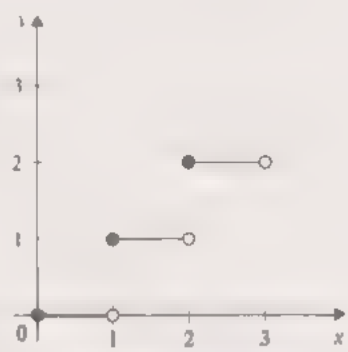
$\left(\text{Hint: Find } \dfrac{f(a+h) - f(a)}{h}\right)$

(ii) Find the derived function of $g: x \longmapsto x$ $(x \in \mathbb{R})$.

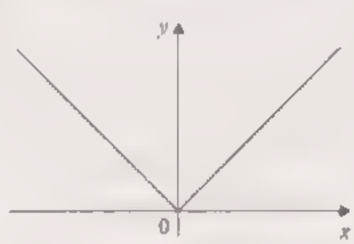
In the next section we shall build on the derived functions that you already know to give systematic techniques for finding derived functions.

Postscript

In the course of this section we have made vital use of two properties of the graphs that we have considered. Firstly, that if the graph of $y = f(x)$ is below $y = x$ at one point, and above at another, then at some point in between, $y = f(x)$ crosses $y = x$. This is fine provided the graph of $y = f(x)$ has no "jumps". Secondly, we have argued that any chord has a parallel tangent attached at a point between the ends of the chord. This is true provided that the graph has no jumps or "corners". The polynomial functions of this section and the sine, cosine, logarithm and exponential functions have neither jumps nor corners. Deciding which functions have neither jumps nor corners is part of a subject called Analysis (the theory of calculus), which is considered in subsequent courses.



The integer part function
A graph with "jumps"



The modulus function
A graph with a "corner"

4.4 DERIVED FUNCTIONS

In this section we continue the task of compiling a list of derived functions and form some conjectures about how the derived function of, say,

$$x \longmapsto 3x^3 - 2x^2 + 5x - 3 \qquad (x \in \mathbb{R})$$

is related to the derived functions of the simpler functions

$$x \longmapsto x^3 \quad (x \in \mathbb{R}), \qquad x \longmapsto x^2 \quad (x \in \mathbb{R}), \quad \text{etc.}$$

We shall make use of the results obtained so far:

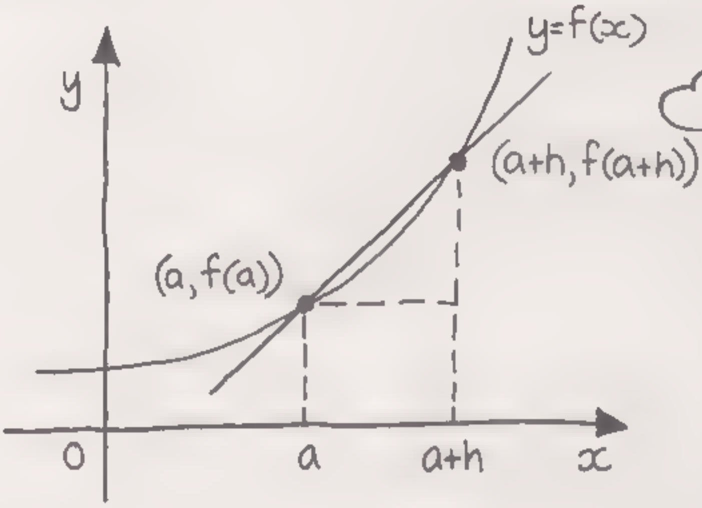
Rule for function f	Rule for derived function f'
$x \longmapsto 1$	$x \longmapsto 0$
$x \longmapsto x$	$x \longmapsto 1$
$x \longmapsto x^2$	$x \longmapsto 2x$
$x \longmapsto x^3$	$x \longmapsto 3x^2$
$x \longmapsto x^3 - x$	$x \longmapsto 3x^2 - 1$
$x \longmapsto \frac{1}{x}$	$x \longmapsto -\frac{1}{x^2}$

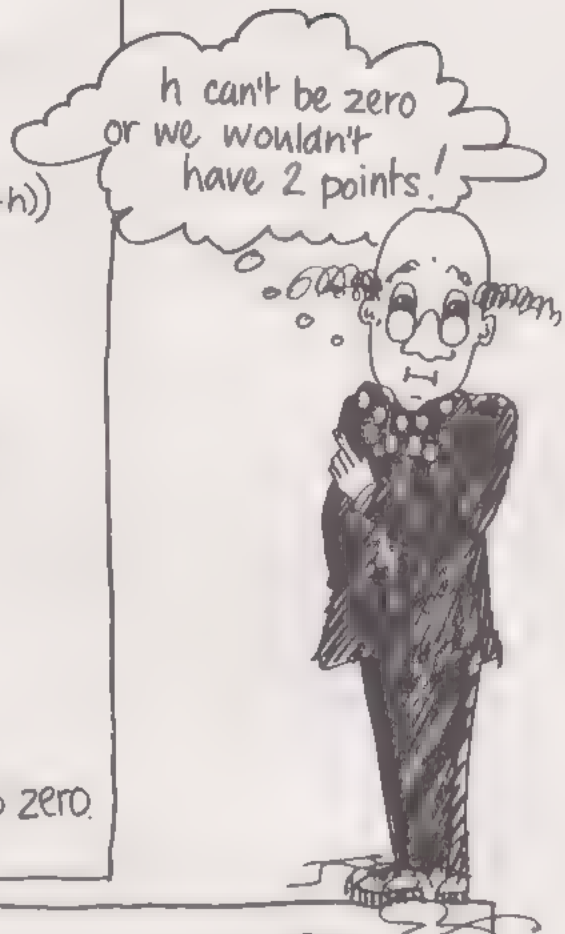
We shall apply the idea of derived function to the finding of horizontal tangents and to the checking of formula iteration for likely convergence.

You will probably need some paper for rough work as well as a pen or pencil.

Now start the tape

1 Reminder

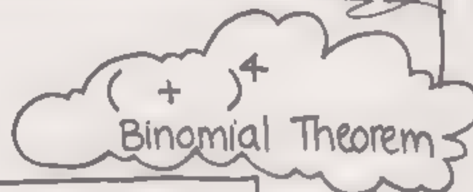




Slope of chord is $\dfrac{f(a+h)-f(a)}{h}$

Slope of tangent: value as h tends to zero.

2 Application to $f(x)=x^4$



$f(a+h) = (a+h)^4 = \boxed{}$

$f(a+h) - f(a) = \boxed{}$

$\dfrac{f(a+h)-f(a)}{h} = \boxed{}$

Slope of tangent $= \boxed{}$

3 Derived Function of $x \mapsto x^4 \; (x \in \mathbb{R})$

Slope of tangent at (a, a^4) is $4a^3$

"Slope function" has rule $a \mapsto 4a^3$

Derived function is $x \mapsto 4x^3 \; (x \in \mathbb{R})$

4 A Problem

Find the derived function for $f : x \mapsto x^5 \; (x \in \mathbb{R})$

$f(a+h) = (a+h)^5 \quad = \quad \boxed{}$

$f(a+h) - f(a) \quad = \quad \boxed{}$

$\dfrac{f(a+h) - f(a)}{h} \quad = \quad \boxed{}$

Slope of tangent at (a, a^5) is $\boxed{}$

Derived function is $x \mapsto \boxed{} \; (x \in \mathbb{R})$

5 Summary and Conjecture

Rule for function	Rule for derived function
$x \mapsto x^1$	$x \mapsto 1$
$x \mapsto x^2$	$x \mapsto 2x$
$x \mapsto x^3$	$x \mapsto 3x^2$
$x \mapsto x^4$	$x \mapsto 4x^3$
$x \mapsto x^5$	$x \mapsto 5x^4$

Conjecture

$x \mapsto x^n$	$x \mapsto \boxed{}$

6 General Result

If $f : x \mapsto x^n \quad (x \in \mathbb{R})$
then $f' : x \mapsto nx^{n-1} \quad (x \in \mathbb{R})$
for <u>any</u> positive integer n.

Is it true for any other values of n?

7 $n = -1$

$\frac{1}{x}$ is x^{-1}

x^n with $n = -1$

General result gives rule for derived function as

$$x \mapsto \underbrace{(-1)}_{n} x^{\overbrace{(-1)-1}^{n-1}}$$

i.e. $x \mapsto -x^{-2}$
or $x \mapsto -\frac{1}{x^2}$, as before.

Must be careful with <u>domains</u>. Neither $x \mapsto \frac{1}{x}$ nor $x \mapsto -\frac{1}{x^2}$ can have zero in the domain.

8 A Rule

We shall assume that the rule for the derived function of

$$x \mapsto x^n$$

is $x \mapsto nx^{n-1}$

for <u>any</u> integer n.

With any necessary care with domains

9 Multiples of Powers of x

Example $f(x) = 4x^2$

$f(a+h) = 4(a+h)^2 = \boxed{}$

$f(a+h) - f(a) = \boxed{}$

$\dfrac{f(a+h)-f(a)}{h} = \boxed{}$

Tangent slope at $(a, 4a^2)$ is $\boxed{}$

so $f'(x) = \boxed{}$

Example $f(x) = -\frac{1}{2}x^2$

$f(a+h) = -\frac{1}{2}(a+h)^2 = \boxed{}$

$f(a+h) - f(a) = \boxed{}$

$\dfrac{f(a+h)-f(a)}{h} = \boxed{}$

Tangent slope at $(a, -\frac{1}{2}a^2)$ is $\boxed{}$

so $f'(x) = \boxed{}$

Conjecture

If $f(x) = Kx^2$
then $f'(x) = \boxed{}$

General Conjecture

If $f(x) = Kx^n$
then $f'(x) = \boxed{}$

10 Result

If $f : x \mapsto Kx^n \quad (x \in \mathbb{R})$
then $f' : x \mapsto Knx^{n-1} \quad (x \in \mathbb{R})$
for n a positive integer,
K a constant.

11 Polynomials

A simple example : $f(x) = 3x^2 + 2x + 1$

$f(a+h) = 3(a+h)^2 + 2(a+h) + 1 = \boxed{}$

$f(a+h) - f(a) = \boxed{}$

$\dfrac{f(a+h) - f(a)}{h} = \boxed{}$

Slope of tangent at $(a, 3a^2+2a+1)$ is $\boxed{}$

$f'(x) = \boxed{}$

12 $f(x) = 3x^2 + 2x + 1$ examined

The "parts" of $f(x)$ are

$3 \times x^2$ $\qquad$ $2 \times x$ $\qquad$ 1

Individually

$3 \times 2x$ $\qquad$ 2×1 $\qquad$ 0

Kx^n gives Knx^{n-1}

Which are the separate parts of

$f'(x) = 6x + 2.$

13 Result

> The derived function for a polynomial function is just "the sum of the separate terms."

Example

Find the derived function of

$$f(x) = 3x^3 + 9x^2 - 3x - 6 \quad (x \in \mathbb{R}).$$

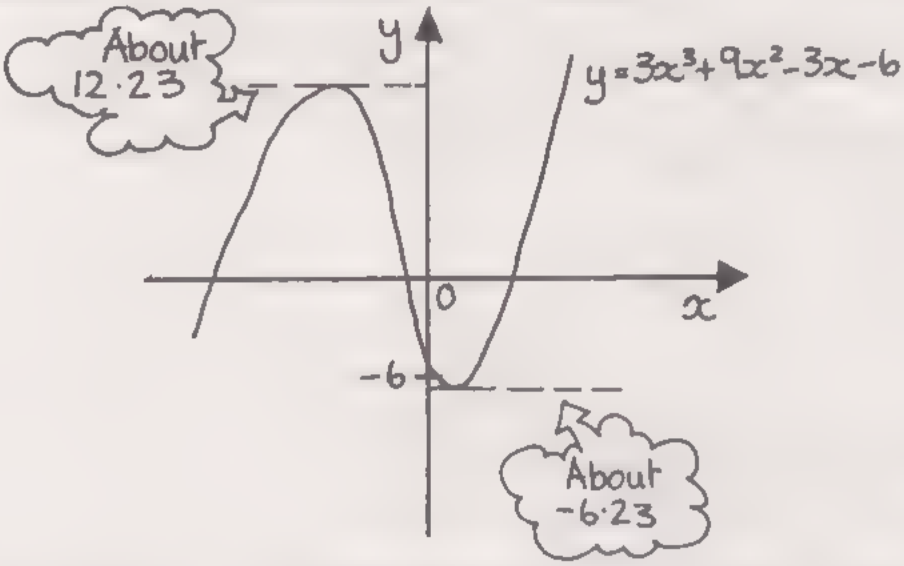
$$f'(x) = 3 \times 3x^2 + 9 \times 2x - 3 \times 1 - 6 \times 0 \quad (x \in \mathbb{R}).$$

$$f'(x) = 9x^2 + 18x - 3 \quad (x \in \mathbb{R}).$$

14 Horizontal Tangents

For $f(x) = 3x^3 + 9x^2 - 3x - 6 \quad (x \in \mathbb{R})$
we have $f'(x) = 9x^2 + 18x - 3 \quad (x \in \mathbb{R})$.

The tangent to the graph of f has slope zero when $x = \boxed{}$ or $x = \boxed{}$.



Horizontal Tangents occur where $f'(x) = 0$.

15 Summary: Horizontal Tangents

To find points with horizontal tangents on the graph of
$y = f(x)$.

1. Write down the derived function as
$f'(x) =$ (expression involving x).
2. Find x-values for which
$f'(x) = 0$.
3. Find corresponding y-values.

16 Example

Find the points with horizontal tangents on the graph of f:

$$y = 2x^3 + 3x^2 - 12x + 1$$

(i.e. $y = f(x)$ for $f(x) = 2x^3 + 3x^2 - 12x + 1$).

1. Write down $f'(x)$: $f'(x) = 2 \times 3x^2 + 3 \times 2x - 12 \times 1 + 0$

   i.e. $f'(x) = 6x^2 + 6x - 12$.

2. Find x-values for which $f'(x) = 0$: $f'(x) = 0$ if and only if $6x^2 + 6x - 12 = 0$.

   i.e. $x^2 + x - 2 = 0$

   i.e. $(x-1)(x+2) = 0$.

   Thus $f'(x) = 0$ for $x = 1$ or $x = -2$

   Check: $f'(1) = 6 \times 1^2 + 6 \times 1 - 12 = 0$

   $f'(-2) = 6 \times (-2)^2 + 6 \times (-2) - 12 = 0$.

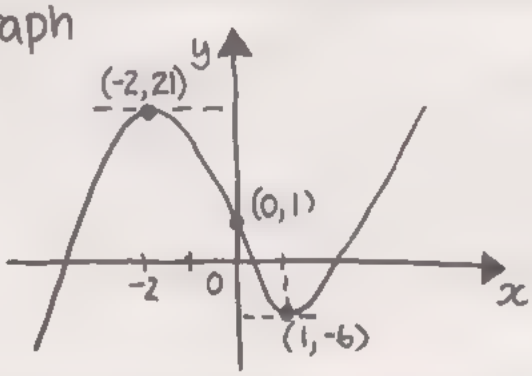
3. Find corresponding y-values:

   when $x = 1$, $y = 2 \times 1^3 + 3 \times 1^2 - 12 \times 1 + 1 = -6$

   when $x = -2$, $y = 2 \times (-2)^3 + 3 \times (-2)^2 - 12 \times (-2) + 1 = 21$

Hence the graph of $y = 2x^3 + 3x^2 - 12x + 1$ has horizontal tangents at $(1, -6)$ and $(-2, 21)$.

17 Sketch Graph



18 Application to Formula Iteration

Consider $x_{n+1} = x_n^3 - x_n^2 + 0{\cdot}5,$
$x_1 = 0{\cdot}5.$

Following the scheme in the TV Section:

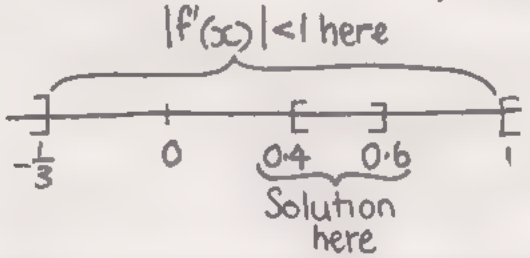
1. Write down the iteration function, $f(x) = x^3 - x^2 + 0{\cdot}5$.
2. Find the corresponding derived function, $f'(x) = 3x^2 - 2x + 0$.
3. Find out where $|f'(x)| < 1$

$$|3x^2 - 2x| < 1$$

	$3x^2 - 2x < 1$	and	$3x^2 - 2x > -1$
i.e.	$1 + 2x - 3x^2 > 0$	and	$3x^2 - 2x + 1 > 0$
i.e.	$(1+3x)(1-x) > 0$	and	$3(x - \frac{2}{6})^2 + \frac{2}{3} > 0$
	$x \in]-\frac{1}{3}, 1[$		always true

4. Locate solution of $x = f(x)$.

Since $x^3 - x^2 + 0{\cdot}5 - x > 0$ when $x = 0{\cdot}4$
and $x^3 - x^2 + 0{\cdot}5 - x < 0$ when $x = 0{\cdot}6$,
there is a solution in $[0{\cdot}4, 0{\cdot}6]$.



This iteration meets the requirements of the test.
$[0{\cdot}4, 0{\cdot}6]$ has enough elbow room in $]-\frac{1}{3}, 1[$.
The iteration will be successful.

Check: calculate $x_2, \ldots, x_{15}$.

18A Answers

$x_1 = 0{\cdot}5$
$x_2 = 0{\cdot}375$
$x_3 = 0{\cdot}4121094$
$x_4 = 0{\cdot}4001561$
$x_5 = 0{\cdot}4039501$
$x_6 = 0{\cdot}4027392$
$x_7 = 0{\cdot}403125$
$x_8 = 0{\cdot}403002$
$x_9 = 0{\cdot}4030412$
$x_{10} = 0{\cdot}4030287$
$x_{11} = 0{\cdot}4030327$
$x_{12} = 0{\cdot}4030314$
$x_{13} = 0{\cdot}4030318$
$x_{14} = 0{\cdot}4030317$
$x_{15} = x_{14}$.

19 Summary

The results we obtained can be summarized as:

$f(x)$	$f'(x)$
x^n	nx^{n-1}
Kx^n	Knx^{n-1}
polynomial	sum of the separate terms

We no longer need to treat *every* function from first principles.

4.5 LIMITS

In this section we shall take a brief look at an idea which has already appeared several times in the course: the idea of a limit. The main purpose of the section is to provide some convenient language and notation for situations that you have already met, although there are some exploratory problems as well.

In Block I, and again in this unit, we have looked at sequences

$$x_1, x_2, x_3, \ldots, x_n, \ldots$$

arising from formula iteration. We know that some formula iterations converge, that is the values

$$x_1, x_2, x_3, \ldots$$

obtained using

$$x_{n+1} = f(x_n)$$

get closer and closer to a number, α, which is a solution of $x = f(x)$. For example, using a calculator and the rule

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \quad x_1 = 1,$$

gives the sequence

$$1, 1.5, 1.4166667, 1.4142157, 1.4142136, 1.4142136, \ldots .$$

We also know that if

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

then

$$2x = x + \frac{2}{x}$$

i.e. $\qquad x = \frac{2}{x}$

i.e. $\qquad x^2 = 2.$

The sequence above certainly seems to be converging rapidly to the solution, $\sqrt{2}$, of this equation. Adapting the language used in Section 4.2, we say that x_n tends to $\sqrt{2}$ as n increases.

What this statement is intended to mean is that we could make the value of x_n as close as we wished to $\sqrt{2}$ by taking sufficiently large values of n (i.e. by doing enough iterations). There is a suggestive notation used in such situations. We write

$$\lim_{n \to \infty} x_n = \sqrt{2},$$

which is read:

"the limit, as n tends to infinity, of x_n is $\sqrt{2}$".

Words of caution are appropriate here. The values of the terms x_n never reach $\sqrt{2}$: they *cannot*, because we started with $x_1 = 1$ (a rational number) and the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

involves only adding, multiplying and dividing. Hence x_2 is also a rational number, likewise x_3, and so on. All the terms in the sequence

$$x_1, x_2, \ldots$$

are rational, whereas $\sqrt{2}$ is not. Equally, n never "reaches" infinity, which is not a number.

Providing these points are borne in mind, the language and notation for limits is very useful.

Another example of a limit is probably familiar to you in a different guise. Consider the statement

$$\tfrac{1}{3} = 0.3333\ldots$$

What does it really mean? If we are to avoid statements about "going on forever", it must mean that

$$0.3 \quad = \frac{3}{10},$$
$$0.33 \quad = \frac{3}{10} + \frac{3}{100},$$
$$0.333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000},$$

etc.,

are better successive approximations to $\frac{1}{3}$ and that we can obtain as good an approximation as we choose by taking enough terms. Using the notation just introduced:

$$\lim_{n \to \infty} \left(\frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n} \right) = \frac{1}{3}.$$

Our third, and final, example is provided by the process of finding derived functions, that has occupied much of this unit. The key step in finding the derived function of f is to investigate

$$\frac{f(a+h) - f(a)}{h}$$

The slope of the chord joining $(a, f(a))$ to $(a+h, f(a+h))$.

as h tends to zero. If $\dfrac{f(a+h) - f(a)}{h}$ tends to some value as h tends to zero, then we have obtained the slope of the tangent to $y = f(x)$ at $(a, f(a))$. Our limit notation can be adapted to this situation:

$$\text{slope of tangent at } (a, f(a)) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Provided that the limit exists.

The right-hand side is read as: "the limit, as h tends to zero, of ... "

The careful and precise definition of what is meant by a limit, and the rigorous finding and justification of limits, belongs to the part of mathematics called Analysis. Such precision and formality are not appropriate to a foundation course. We shall make some reasonable assumptions and investigate one or two useful limits in a fairly informal way.

Our basic assumption is

$$\lim_{n \to \infty} \frac{1}{n} = 0,$$

i.e. we may make $\frac{1}{n}$ as close to zero as we please by taking sufficiently large values of n. Most of our limits will be related, more or less directly, to this one.

Note: We do not say $\frac{1}{n}$ gets *smaller* because -1000 is less than (i.e. smaller than) $\frac{1}{100}$ but -1000 is farther from zero than is $\frac{1}{100}$.

Problem 4.5.1

(i) (Preliminaries) Suppose that a and b are real numbers such that $0 < a < b$. Show that $0 < \frac{1}{b} < \frac{1}{a}$.

(ii) Use the Binomial Theorem to show that

$$2^n > n$$

for any positive integer n. [Hint: $2 = 1 + 1$.]

(iii) (a) Deduce that $0 < \frac{1}{2^n} < \frac{1}{n}$ for any positive integer n.

(b) Deduce that $\lim_{n \to \infty} \frac{1}{2^n} = 0$.

The idea behind Solution 4.5.1 is fundamental to working with limits. The limit that you are trying to find (of $\frac{1}{2^n}$ in Solution 4.5.1) is compared with one already known $\left(\text{of } \frac{1}{n}\right)$.

Although we shall not do so here, the result of Solution 4.5.1,

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = \lim_{n\to\infty}\frac{1}{2^n} = 0,$$

can be generalized to

$$\lim_{n\to\infty} r^n = 0 \quad \text{for any real number } r \text{ with } -1 < r < 1.$$

i.e. $|r| < 1$

Although we shall not justify this result, you may use it when necessary.

The result $\lim_{n\to\infty} r^n = 0$ for $-1 < r < 1$ can be applied to show why $\frac{1}{3} = 0.333\ldots$ is, indeed, a reasonable statement. If we look at the successive approximations to $\frac{1}{3}$,

$$s_1 = 0.3 \qquad = 3\left(\frac{1}{10}\right)$$

We have used $s_1, s_2, \ldots$ because the approximations are *sums* of terms.

$$s_2 = 0.33 \qquad = 3\left(\frac{1}{10}+\frac{1}{100}\right) = 3\left(\frac{1}{10}+\frac{1}{10^2}\right)$$

$$s_3 = 0.333 \qquad = 3\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}\right)$$

$$s_n = 0.\underbrace{33\ldots3}_{n\text{ terms}} = 3\left(\frac{1}{10}+\frac{1}{10^2}+\cdots+\frac{1}{10^n}\right),$$

we see that each successive s_i involves one more term.

It is possible to obtain a formula for s_n, as follows:

$$s_n = 3\left(\frac{1}{10}+\frac{1}{10^2}+\cdots+\frac{1}{10^n}\right),$$

$$\frac{1}{10}s_n = 3\left(\frac{1}{10^2}+\frac{1}{10^3}+\cdots+\frac{1}{10^{n+1}}\right);$$

subtracting:

$$s_n - \frac{1}{10}s_n = 3\left(\frac{1}{10}\right) - 3\left(\frac{1}{10^{n+1}}\right),$$

all other terms cancelling. Thus

$$\frac{9}{10}s_n = 3\left(\frac{1}{10}-\left(\frac{1}{10}\right)^{n+1}\right),$$

i.e.

$$s_n = \frac{10}{9}\times 3\left(\frac{1}{10}-\left(\frac{1}{10}\right)^{n+1}\right)$$

$$= \frac{10}{3}\left(\frac{1}{10}-\left(\frac{1}{10}\right)^{n+1}\right).$$

By taking n large enough, we can make the $\left(\frac{1}{10}\right)^{n+1}$ term as close to zero as we please. So

$$\lim_{n\to\infty} s_n = \frac{10}{3}\left(\frac{1}{10}\right)$$

$$= \frac{1}{3}$$

The argument used above to find s_n will generalize. Suppose we have a sequence constructed as follows:

$$s_1 = a$$

$$s_2 = a + ar$$

$$s_3 = a + ar + ar^2$$

$$\vdots$$

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$\vdots$$

Our example obtained from $\frac{1}{3}$ corresponds to $a = \frac{3}{10}$, $r = \frac{1}{10}$

where a, r are two real numbers.

Problem 4.5.2

(i) Write down $s_n - rs_n$, and hence show that

$$s_n = \frac{a}{1-r}(1-r^n), \quad \text{provided } r \neq 1.$$

(ii) Deduce that, provided $-1 < r < 1$,

$$\lim_{n \to \infty} s_n = \frac{a}{1-r}$$

The "infinite sum"

$$a + ar + ar^2 + ar^3 + \cdots$$

is known as the *geometric series* with *first term a* and *common ratio r*. (The ratio of each term to the previous one is r.) The result of Solution 4.5.2 is sometimes referred to as "the sum to infinity" of this geometric series, i.e.

$$a + ar + ar^2 + \cdots = \frac{a}{1-r} \quad \text{when } -1 < r < 1.$$

Problem 4.5.3

Interpret 0.111... as a geometric series, and hence find its value as a rational number.

Any recurring decimal can be dealt with in a similar way. For example, we may interpret

0.13131313...

as the geometric series

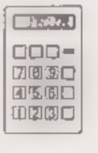
$$\frac{13}{100} + \frac{13}{10000} + \frac{13}{1000000} + \cdots$$

with first term $a = \frac{13}{100}$ and common ratio $r = \frac{1}{100}$. Since $r = \frac{1}{100}$ does satisfy $-1 < r < 1$, we can deduce that

$$0.13131313\ldots = \frac{a}{1-r} = \frac{\frac{13}{100}}{1-\frac{1}{100}} = \frac{\frac{13}{100}}{\frac{99}{100}} = \frac{13}{99}$$

We conclude this brief introduction to the language and notation of limits by asking you to explore one limit that will be useful later, in Block III.

Problem 4.5.4



(i) A certain bank pays 10% interest per annum, but adds interest half-yearly to the account. If £1 is deposited on 1 January, how much is in the account on 31 December of the same year, providing no withdrawal is made?

(ii) A rival bank also pays 10% per annum, but pays interest quarterly. Repeat part (i) for this case.

(iii) What is the effect of a bank keeping the rate at 10% per annum, but paying interest 5, 6, 7,... times per year?

The results of Solution 4.5.4 suggest that the ultimate in "friendly bank managers", who offered to pay interest as often as the customer chose to

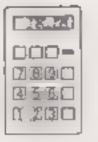
nominate, would never give the customer more than

$$\lim_{n \to \infty} £\left(1 + \frac{0.1}{n}\right)^n \simeq £1.1052$$

in the account at the end of the year.

A closely related limit is

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$



Problem 4.5.5

Find an approximate value for

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

by taking large values of n. Compare your result with inverse ln of 1.

As suggested by Solution 4.5.5, the result

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e,$$

the base for the logarithms produced by the ln button, is actually true, and we shall make some use of the fact in Block III.

OBJECTIVES

After studying this unit you should be able to:

1 draw cobweb and staircase diagrams to illustrate formula iteration;

2 find, in simple cases, derived functions by applying the limit of chord slope method, i.e.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h};$$

3 write down the derived function of a given polynomial function;

4 use derived functions to obtain the equations of tangents to curves and to locate horizontal tangents;

5 use the derived function to determine whether a given rule for formula iteration will converge;

6 convert a given recurring decimal into a rational number using the formula $\dfrac{a}{1-r}$ for the geometric series

$$a + ar + ar^2 + \cdots, \quad |r| < 1$$

SOLUTIONS TO PROBLEMS IN THE TEXT

Solution 4.1.1

(i) (a) $x_1 = 3.5$
$x_2 = 4.25$
$x_3 \simeq 7.06 \quad (7.0625)$
$x_4 \simeq 27.63 \quad (27.628906)$
$x_5 \simeq 658.84 \quad (658.84084)$
$x_6 \simeq 431441.88$
$x_7 \simeq 1.8614 \times 10^{11}$

(b) $x_1 = 1.5$
$x_2 = 2.25$
$x_3 \simeq 2.06 \quad (2.0625)$
$x_4 \simeq 2.00 \quad (2.0039063)$
$x_5 \simeq 2.00 \quad (2.0000153)$
$x_6 = 2$
$x_7 = 2$

Note: We have given the full displayed values for reference

(ii) Equation is

$$x = x^2 - 4x + 6,$$

i.e. $x^2 - 5x + 6 = 0.$

(iii) Factorizing:

$$(x-2)(x-3) = 0,$$

hence $x - 2 = 0$ or $x - 3 = 0,$
thus $x = 2$ or $x = 3.$
Solutions are 2 and 3.

Solution 4.1.2

Our displayed values were:

(i) $x_1 = 2.9$
$x_2 = 2.81$
$x_3 = 2.6561$
$x_4 = 2.4304672$
$x_5 = 2.185302$
$x_6 = 2.0343368$
$x_7 = 2.001179$

The iteration is converging to the solution $x = 2$.

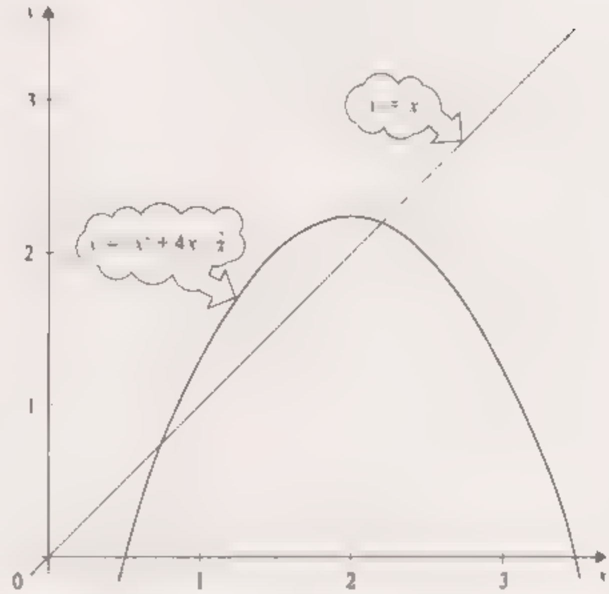
(ii) $x_1 = 3.1$
$x_2 = 3.21$
$x_3 = 3.4641$
$x_4 = 4.1435888$
$x_5 = 6.594973$
$x_6 = 23.113777$
$x_7 = 447.79157$

The iteration diverges.

Solution 4.1.3

(i) In completed square form,

$$y = -x^2 + 4x - \tfrac{7}{4} = -(x-2)^2 + \tfrac{9}{4}.$$



(ii) At the intersections of the graphs we have

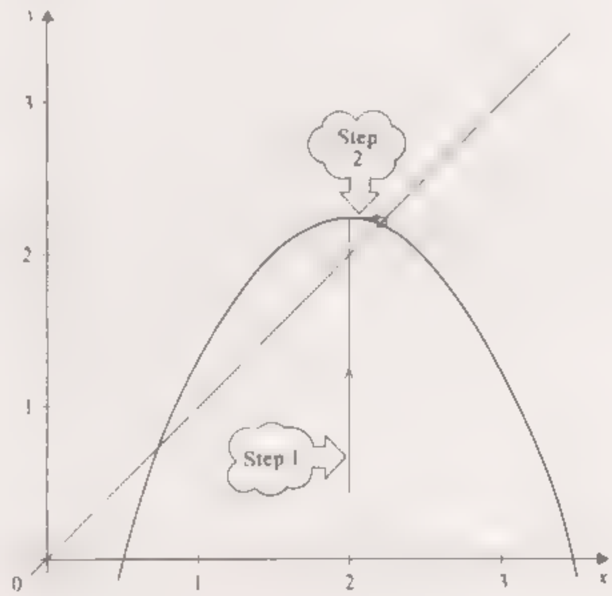
$$y = x \quad \textit{and} \quad y = -x^2 + 4x - \tfrac{7}{4},$$

so the x coordinates satisfy

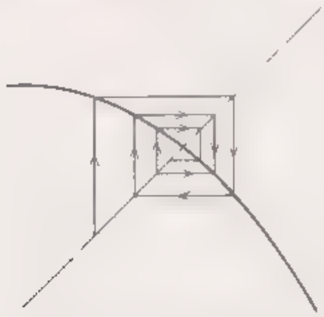
$$x = -x^2 + 4x - \tfrac{7}{4}$$

i.e. $x^2 - 3x + \tfrac{7}{4} = 0.$

(iii) (a)

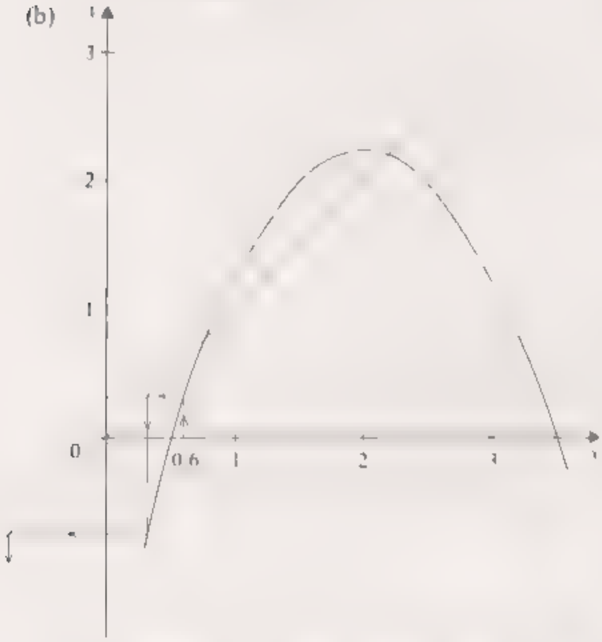


The conclusion is that the iteration converges to the larger of the two solutions of the equation in part (ii). An enlarged view of the region around the intersection would look as follows:



For obvious reasons, such diagrams are called "cobweb" diagrams.

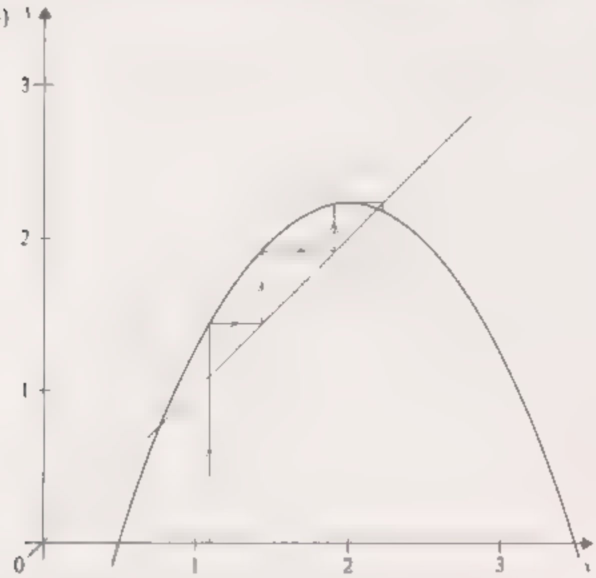
(b)



The conclusion is that the iteration diverges. This type of diagram is often referred to as a "staircase" diagram.

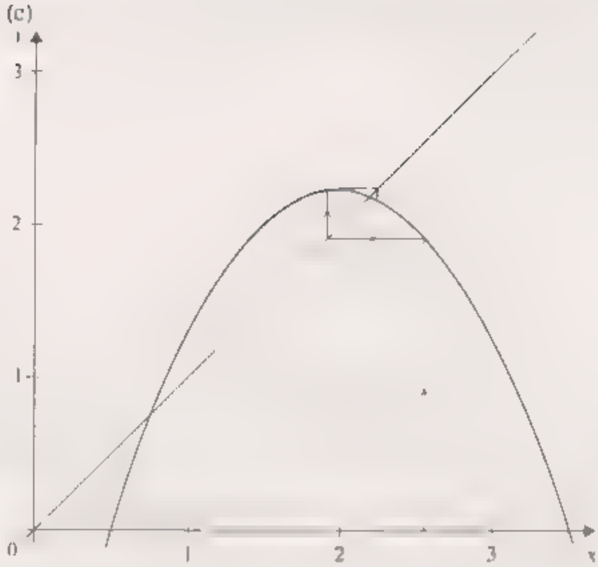
(iv) (a) By drawing a diagram like that in part (iii)(b), we see that if we start to the left of the smaller solution, the iteration will diverge

(b)



The iteration converges to the larger solution

(c)

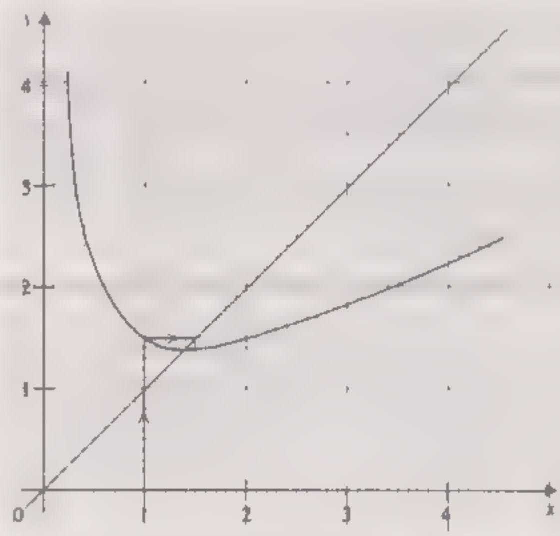


The iteration converges to the larger solution.

Solution 4.1.4

(i) (a) From the graph, α is approximately 1.4.

(b)



The iteration converges to α.

(ii) (a) The slope of the chord joining (x_1, x_2) and (α, α) is

$$\frac{\alpha - x_2}{\alpha - x_1}$$

Now $x_1 = 1$, so $x_2 = \frac{1}{2}(1 + \frac{2}{1}) = \frac{3}{2}$.

Hence the slope is $\dfrac{\alpha - \frac{3}{2}}{\alpha - 1}$.

Since $\alpha \simeq 1.4$, this slope is approximately

$$\frac{1.4 - 1.5}{1.4 - 1}$$

i.e. chord slope $\simeq -0.25$.

Thus $|\text{chord slope}| \simeq 0.25$

(b) The slope of the chord joining (x_2, x_3) and (α, α) is $\dfrac{\alpha - x_3}{\alpha - x_2}$. Since $x_2 = \dfrac{3}{2}$, $x_3 = \dfrac{1}{2}\left(\dfrac{3}{2} + \dfrac{2}{3/2}\right) = \dfrac{17}{12}$.

Using $\alpha \simeq 1.4$, we have

$$\text{chord slope} \simeq \frac{1.4 - (17/12)}{1.4 - 1.5} = \frac{1}{6}.$$

Thus $|\text{chord slope}| \simeq \dfrac{1}{6}$.

(c) Slope of chord joining (x_n, x_{n+1}) and (α, α) is $\dfrac{\alpha - x_{n+1}}{\alpha - x_n}$. Putting $\alpha = \dfrac{1}{2}\left(\alpha + \dfrac{2}{\alpha}\right)$, and $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$ in the numerator

$$\frac{\alpha - x_{n+1}}{\alpha - x_n} = \frac{\frac{1}{2}\left(\alpha + \frac{2}{\alpha}\right) - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)}{\alpha - x_n}$$

$$= \frac{1}{2}\left(\frac{\alpha + \frac{2}{\alpha} - x_n - \frac{2}{x_n}}{\alpha - x_n}\right)$$

$$= \frac{1}{2}\left(\frac{\alpha - x_n + \frac{2}{\alpha} - \frac{2}{x_n}}{\alpha - x_n}\right)$$

$$= \frac{1}{2}\left(1 + \frac{2}{\alpha - x_n}\left(\frac{1}{\alpha} - \frac{1}{x_n}\right)\right)$$

$$= \frac{1}{2}\left(1 + \frac{2}{\alpha - x_n}\left(\frac{x_n - \alpha}{\alpha x_n}\right)\right)$$

$$= \frac{1}{2}\left(1 - \frac{2}{\alpha x_n}\right)$$

Thus $|\text{chord slope}| = \left|\dfrac{1}{2} - \dfrac{1}{\alpha x_n}\right|$

Now, $\alpha \simeq 1.4$, so if x_n is close to α we also have $x_n \simeq 1.4$. Thus, approximately, this modulus is

$$\left|\frac{1}{2} - \frac{1}{1.4 \times 1.4}\right| \simeq |-0.0102041|$$
$$\simeq 0.0102041.$$

Solution 4.2.1

Following the method used in the text, the equation of a line with slope m has the form

$$y = mx + c$$

for some value of c. This line is to pass through $(3, 9)$, so we must have

$$9 = m \times 3 + c$$
i.e. $c = 9 - 3m$

The equation of the line through $(3, 9)$ with slope m is thus $y = mx + (9 - 3m)$. This line intersects the curve $y = x^2$ where

$$x^2 = mx + 9 - 3m,$$
i.e. $x^2 - mx + 3m - 9 = 0$.

We know that one intersection is at $x = 3$, so the equation must factorize as

$$(x - 3)(x \ldots) = 0.$$

To make the constant term, $3m - 9$, correct, the factorization must be

$$(x - 3)(x - m + 3) = 0$$

The two points of intersection are at $x = 3$ and $x = m - 3$. If these points are to be the same, we require

$$3 = m - 3,$$
i.e. $m = 6$.

Thus, the tangent to $y = x^2$ at $(3, 9)$ has slope 6 and the equation is

$$y = mx + 9 - 3m$$
$$= 6x + 9 - 18$$
$$= 6x - 9,$$
i.e. $y = 6x - 9$.

Solution 4.2.2

(i) (a) The x coordinate at $(-1, 1)$ is -1, so the slope of the tangent is $2 \times (-1) = -2$.

(b) The slope of the tangent at $(\frac{1}{2}, \frac{1}{4})$ is $2 \times (\frac{1}{2}) = 1$

(ii) (a) We want the equation of a line through $(-1, 1)$ with slope -2. The equation must be of the form $y = (-2)x + c$, for some number c. The line is to pass through $(-1, 1)$, so

$$1 = (-2) \times (-1) + c,$$

hence $c = -1$. The equation is

$$y = -2x - 1.$$

(b) By a similar argument to part (ii)(a), the equation is of the form $y = 1 \times x + c$. Since the line passes through $(\frac{1}{2}, \frac{1}{4})$, we have $\frac{1}{4} = \frac{1}{2} + c$, so $c = -\frac{1}{4}$. The required equation is

$$y = x - \tfrac{1}{4}$$

N.B. From now on, we shall concentrate on the *slopes* of the tangents to a curve, where the equation of a particular tangent is required, we shall usually work it out from first principles, using the slope and the fact that the tangent passes through the point on the curve.

Solution 4.2.3

(i) The line has slope m so it has an equation of the form $y = mx + c$. The line has to pass through $(a, 4a - a^2)$, so we have

$$4a - a^2 = ma + c$$
i.e. $c = 4a - a^2 - ma$.

Hence the line has equation

$$y = mx + 4a - a^2 - ma.$$

(ii) At the points of intersection we have

$$y = 4x - x^2 \quad \text{and} \quad y = mx + 4a - a^2 - ma.$$

Hence the x coordinates of the points of intersection satisfy

$$4x - x^2 = mx + 4a - a^2 - ma,$$
i.e. $x^2 + mx - 4x + 4a - a^2 - ma = 0$,

i.e. $x^2 + (m - 4)x + 4a - a^2 - ma = 0$.

(iii) We know that one point of intersection is where $x = a$, so the quadratic equation in part (ii) must factorize as

$$(x - a)(x \ldots) = 0.$$

In order to get the constant term, $4a - a^2 - ma$, correct, the factorization must be

$$(x - a)(x - 4 + a + m) = 0$$

Thus, the intersections occur where $x = a$ and where $x = 4 - a - m$. If these two values are to be the same, then

$$a = 4 - a - m$$
i.e. $m = 4 - 2a$

(iv) (a) The slope of the tangent to $y = 4x - x^2$ at $x = a$ is $4 - 2a$.

(b) The derived function (i.e. the "slope-function") has rule $a \longmapsto 4 - 2a$, or $x \longmapsto 4 - 2x$, so the derived function is

$$g' : x \longmapsto 4 - 2x \qquad (x \in \mathbb{R}).$$

Solution 4.2.4

(i) The coordinates of $(a + h, g(a + h))$ are $(a + h, (a + h)^3)$, i.e. $(a + h, a^3 + 3a^2h + 3ah^2 + h^3)$.

(ii) The slope of the chord is

$$\frac{g(a+h) - g(a)}{h} = \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} = \frac{3a^2h + 3ah^2 + h^3}{h} = 3a^2 + 3ah + h^2.$$

(iii) As h tends to zero, $3a^2$ remains the same and both $3ah$ and h^2 tend to zero. Hence the chord slope tends to $3a^2$.

(iv) The slope of the tangent at (a, a^3) on the graph of $y = x^3$ is $3a^2$.

(v) The derived function g' of g is

$$g' : x \longmapsto 3x^2 \qquad (x \in \mathbb{R}).$$

Solution 4.2.5

The graph of $y = x^3$ will have a horizontal tangent if the slope of the tangent is zero. The slope of the tangent at (x, x^3) is $3x^2$; this slope is zero when $x = 0$. Hence the graph has just *one* horizontal tangent: at the origin, $(0, 0)$.

Solution 4.2.6

Following the scheme in Solution 4.2.4:

Coordinates of $(a + h, k(a + h))$ are

$$(a + h, (a + h)^3 + (a + h)),$$

i.e. $(a + h, a^3 + 3a^2h + 3ah^2 + h^3 + a + h)$.

Slope of chord joining

$(a, a^3 + a)$ to $(a + h, (a + h)^3 + (a + h))$ is

$$\frac{k(a+h) - k(a)}{h} = \frac{a^3 + 3a^2h + 3ah^2 + h^3 + a + h - a^3 - a}{h}$$

$$= \frac{3a^2h + 3ah^2 + h^3 + h}{h}$$

$$= 3a^2 + 3ah + h^2 + 1$$

$$= (3a^2 + 1) + 3ah + h^2.$$

As h tends to zero, this chord slope tends to $3a^2 + 1$. Hence, the slope of the tangent at $(a, a^3 + a)$ on the graph of $y = x^3 + x$ is $3a^2 + 1$. The derived function is $k' : x \longmapsto 3x^2 + 1$ $(x \in \mathbb{R})$.

The curve will have a horizontal tangent at any point whose x coordinate satisfies $3x^2 + 1 = 0$. Since x^2 cannot be negative for $x \in \mathbb{R}$, there are no such points. Hence the graph of $y = x^3 + x$ has *no* horizontal tangents.

Solution 4.3.1

(i) From $x = x^3 - \frac{11}{16}x + \frac{3}{4}$ we have

$$0 = x^3 - \frac{11}{16}x + \frac{3}{4} - x,$$

i.e.

$$0 = x^3 - \frac{27}{16}x + \frac{3}{4}.$$

(ii) (a) $x_1 = 0.6$
$x_2 = 0.5535$
$x_3 = 0.5390403$
$x_4 = 0.5360357$
$x_5 = 0.5354969$
$x_6 = 0.5354033$
$x_7 = 0.5353872$
$x_8 = 0.5353844$

Thus $x_7 = x_8 = 0.5354$ to 4 decimal places. The sequence converges.

(b) $x_1 = 1$
$x_2 = 1.0625$
$x_3 = 1.2189941$
$x_4 = 1.7233019$
$x_5 = 4.6830389$
$x_6 = 100.23345$
$x_7 = 1006951.7$
$x_8 = 1021 \times 10^{18}$

The sequence diverges.

Solution 4.3.2

(i) $$f(a + h) = (a + h)^3 - \frac{11}{16}(a + h) + \frac{3}{4}$$

$$= a^3 + 3a^2h + 3ah^2 + h^3 - \frac{11}{16}a - \frac{11}{16}h + \frac{3}{4};$$

$$f(a) = a^3 - \frac{11}{16}a + \frac{3}{4}.$$

Hence

$$f(a + h) - f(a) = 3a^2h + 3ah^2 + h^3 - \frac{11}{16}h.$$

So

$$\frac{f(a+h) - f(a)}{h} = 3a^2 + 3ah + h^2 - \frac{11}{16}$$

(since $h \neq 0$)

$$= 3a^2 - \frac{11}{16} + h(3a + h).$$

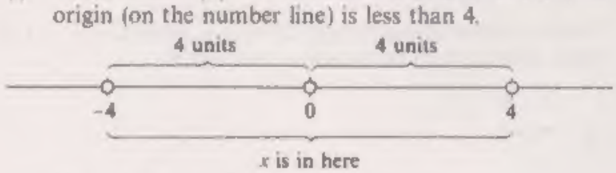
This approaches $3a^2 - \frac{11}{16}$ as h approaches zero. Thus, the slope of the tangent to $y = f(x)$ at the point $(a, f(a))$ is $3a^2 - \frac{11}{16}$.

(ii) The rule is $a \longmapsto 3a^2 - \frac{11}{16}$, or, as usually written, $x \longmapsto 3x^2 - \frac{11}{16}$; i.e. $f' : x \longmapsto 3x^2 - \frac{11}{16}$, or, alternatively,

$$f'(x) = 3x^2 - \frac{11}{16}.$$

Solution 4.3.3

(i) The statement $|x| < 4$ asserts that distance of x from the origin (on the number line) is less than 4.



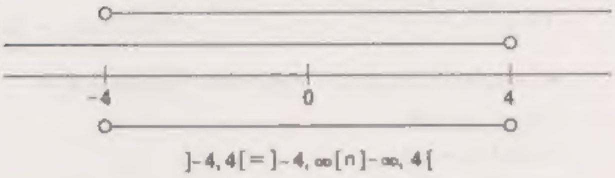
The solution set is $]-4, 4[$.

(ii) The statement $x \in]-4,4[$ is equivalent to two inequalities:

$$-4 < x < 4,$$

or, alternatively,

$$-4 < x \quad \text{and} \quad x < 4.$$



Solution 4.3.4

Following the hint, the distance of $x - 1$ from the origin is less than 3 when

$$-3 < x - 1 < 3.$$

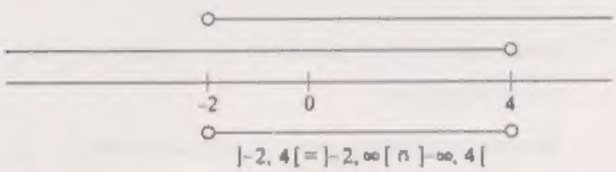
This is equivalent to the two inequalities

$$-3 < x - 1 \quad \text{and} \quad x - 1 < 3.$$

Using the definition of $<$, these become

$$(x-1)-(-3) > 0 \quad \text{and} \quad 3-(x-1) > 0,$$

i.e. $$x + 2 > 0 \quad \text{and} \quad 4 - x > 0.$$



Thus the solution set is $]-2,4[$.

Solution 4.3.5

We may rewrite $\left|3x^2 - \frac{11}{16}\right| < 1$ as

$$-1 < 3x^2 - \frac{11}{16} < 1,$$

giving

$$-1 < 3x^2 - \frac{11}{16} \quad \text{and} \quad 3x^2 - \frac{11}{16} < 1,$$

i.e.

$$3x^2 + \frac{5}{16} > 0 \quad \text{and} \quad -3x^2 + \frac{27}{16} > 0.$$

The left-hand inequality is true for all values of x (i.e. has solution set $\mathbb{R}$) because $3x^2 \geq 0$ and hence $3x^2 + \frac{5}{16} > 0$.

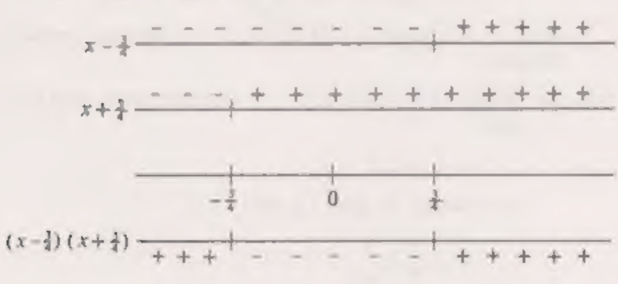
Factorizing the right-hand inequality gives

$$-3(x^2 - \tfrac{9}{16}) > 0$$

i.e. $-3(x - \frac34)(x + \frac34) > 0$.

Now, -3 is negative so, to obtain a positive product, we require

$$(x - \tfrac34)(x + \tfrac34) < 0.$$



Thus, the solution set for the second inequality is $]-\frac34, \frac34[$.

The solution of $\left|3x^2 - \frac{11}{16}\right| < 1$ is $\mathbb{R} \cap]-\frac34, \frac34[$, i.e. $]-\frac34, \frac34[$.

Solution 4.3.6

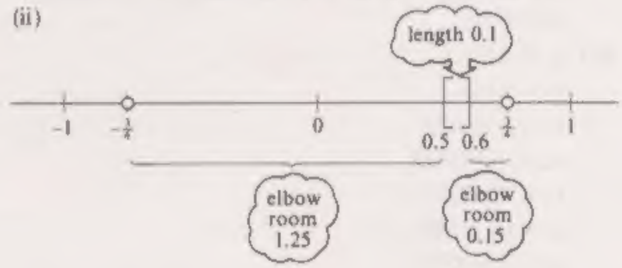
(i) When $x = 0.5$, $x^3 - \frac{11}{16}x + \frac34 = 0.53125$, which is greater than 0.5.

When $x = 0.6$, $x^3 - \frac{11}{16}x + \frac34 = 0.5535$, which is less than 0.6.

Hence, the graph of $y = x^3 - \frac{11}{16}x + \frac34$ lies above that of $y = x$ when $x = 0.5$ and below it when $x = 0.6$. Thus $x = x^3 - \frac{11}{16}x + \frac34$ has a solution in $[0.5, 0.6]$.

Alternatively, as in Block I, *Unit 1*, you could check that $x^3 - \frac{27}{16}x + \frac34$ changes sign in $[0.5, 0.6]$, which is what we shall do in future.

(ii)



The elbow room for $[0.5, 0.6]$ is at least as long as the interval, so the iteration will converge for any starting value in $[0.5, 0.6]$.

Solution 4.3.7

(i) $|f'(x)| < 1$ is the same as

$$-1 < 3x^2 + \tfrac14 < 1.$$

Firstly, consider $-1 < 3x^2 + \frac14$, i.e.

$$3x^2 + \tfrac14 - (-1) > 0,$$

i.e. $3x^2 + \frac54 > 0$.

This is true for all x, i.e. the solution set is $\mathbb{R}$.

Next, $3x^2 + \frac14 < 1$, i.e.

$$1 - (3x^2 + \tfrac14) > 0,$$

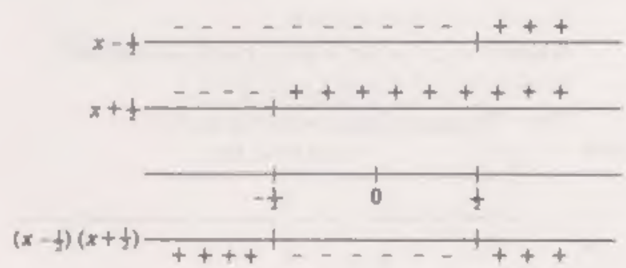
i.e. $-3x^2 + \frac34 > 0$,

i.e. $(-3)(x^2 - \frac14) > 0$,

i.e. $(-3)(x - \frac12)(x + \frac12) > 0$.

Arguing as before, we need

$$(x - \tfrac12)(x + \tfrac12) < 0.$$



The solution set is $]-\frac12, \frac12[$.

Hence, $|f'(x)| < 1$ for $\mathbb{R} \cap]-\frac12, \frac12[$, i.e. $]-\frac12, \frac12[$.

(ii) (a) We have $x = x^3 + \frac{1}{4}x + \frac{1}{8}$, so

$$0 = x^3 + \tfrac{1}{4}x + \tfrac{1}{8} - x,$$

i.e.

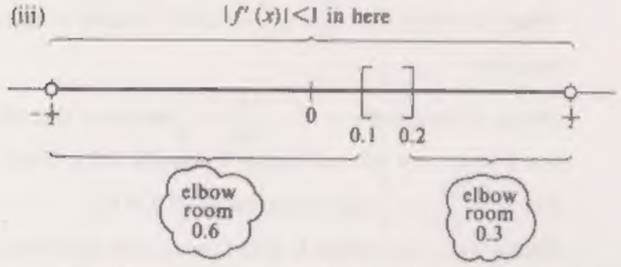
$$x^3 - \tfrac{3}{4}x + \tfrac{1}{8} = 0.$$

(b) When $x = 0.1$, $x^3 - \frac{3}{4}x + \frac{1}{8} = 0.051$.

When $x = 0.2$, $x^3 - \frac{3}{4}x + \frac{1}{8} = -0.017$.

Thus $x^3 - \frac{3}{4}x + \frac{1}{8}$ changes sign in $[0.1, 0.2]$ and hence the equation has a solution in $[0.1, 0.2]$.

(iii)



There is a solution, α, in $[0.1, 0.2]$; the interval $[0.1, 0.2]$ is entirely within $[-\frac{1}{2}, \frac{1}{2}]$ and has elbow room on each side which is at least the length (0.1) of the interval. Thus, any starting value in $[0.1, 0.2]$ will give a convergent sequence.

(iv) $x_1 = 0.1$

$x_2 = 0.151$

$x_3 = 0.166193$

$x_4 = 0.1711385$

$x_5 = 0.172797$

$x_6 = 0.1733588$

$x_7 = 0.1735497$

$x_8 = 0.1736147$

$x_9 = 0.1736368$

Hence $\alpha = 0.1736$, to 4 decimal places.

Solution 4.3.8

(i) From $|3x^2 - 6x + \frac{8}{3}| < 1$ we obtain (as before) $-1 < 3x^2 - 6x + \frac{8}{3}$ and $3x^2 - 6x + \frac{8}{3} < 1$. Firstly, we need $3x^2 - 6x + \frac{11}{3} > 0$. Using the "completed square" form: $3(x-1)^2 + \frac{2}{3} > 0$. Since a square is non-negative, this inequality is satisfied for all $x \in \mathbb{R}$.

Next we need $1 - (3x^2 - 6x + \frac{8}{3}) > 0$, i.e.

$$-3x^2 + 6x - \tfrac{5}{3} > 0$$

or $\quad -3(x^2 - 2x + \frac{5}{9}) > 0$.

Using the hint, this is

$$-3(x - \tfrac{1}{3})(x - \tfrac{5}{3}) > 0.$$

Using the same methods as in previous solutions, this has solution set $]\frac{1}{3}, \frac{5}{3}[$.

Thus $|f'(x)| < 1$ has solution set $\mathbb{R} \cap \,]\frac{1}{3}, \frac{5}{3}[\, = \,]\frac{1}{3}, \frac{5}{3}[$.

(ii) The equation $x = f(x)$ can be written

$$0 = x^3 - 3x^2 + \tfrac{8}{3}x + \tfrac{1}{6} - x,$$

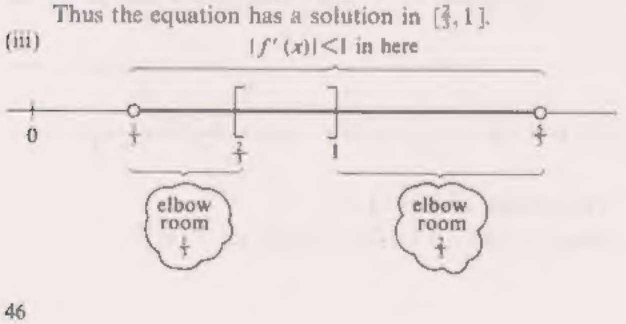
i.e. $x^3 - 3x^2 + \frac{5}{3}x + \frac{1}{6} = 0$.

When $x = \frac{2}{3}$, $x^3 - 3x^2 + \frac{5}{3}x + \frac{1}{6} = \frac{13}{54} \simeq 0.2407407$.

When $x = 1$, $x^3 - 3x^2 + \frac{5}{3}x + \frac{1}{6} = -\frac{1}{6}$.

Thus the equation has a solution in $[\frac{2}{3}, 1]$.

(iii)



Since there is sufficient elbow room, namely $\frac{1}{3}$, iteration with any starting value in $[\frac{2}{3}, 1]$ will converge. Since $0.7 \in [\frac{2}{3}, 1]$, iteration with $x_1 = 0.7$ will converge.

Solution 4.3.9

(i) $f(a+h) = 1$, because f(anything) is 1.

Thus $\dfrac{f(a+h) - f(a)}{h} = \dfrac{1-1}{h} = \dfrac{0}{h} = 0$ (because $h \neq 0$).

As h approaches 0, this expression "approaches" 0.

Thus $f'(a) = 0$.

Thus $f' : a \longmapsto 0$,

or $\quad f' : x \longmapsto 0, \quad x \in \mathbb{R}$.

Remark: The above argument looks the same if $f(x) =$ any constant, which indicates why constants in f disappear in f'.

(ii) Since $g(a+h) = a + h$ and $g(a) = a$, we have

$$\frac{g(a+h) - g(a)}{h} = \frac{a + h - a}{h} = \frac{h}{h} = 1 \ (h \neq 0).$$

As h approaches 0, this expression approaches 1. Thus $g'(a) = 1$, so $g' : x \longmapsto 1$, $x \in \mathbb{R}$.

Solution 4.5.1

(i) We are asked to establish that $0 < \dfrac{1}{b} < \dfrac{1}{a}$. We are given that a and b are both positive ($0 < a < b$). Hence $\dfrac{1}{b}$ and $\dfrac{1}{a}$ are both positive.

In order to establish $\dfrac{1}{b} < \dfrac{1}{a}$, we must show that $\dfrac{1}{a} - \dfrac{1}{b}$ is positive, i.e. that $\dfrac{b-a}{ab}$ is positive, and this we can do because $a < b$ so $b - a$ is positive and ab is positive because a and b are positive.

Having explored the situation, we can write out a justification.

Since $a < b$, $b - a$ is positive.

Since $0 < a < b$, a and b are positive, thus ab is positive.

Hence $\dfrac{b-a}{ab}$ is positive.

Thus $\dfrac{1}{a} - \dfrac{1}{b}$ is positive, i.e. $\dfrac{1}{b} < \dfrac{1}{a}$.

But a, b are positive so $\dfrac{1}{a}, \dfrac{1}{b}$ are positive.

Hence $0 < \dfrac{1}{b} < \dfrac{1}{a}$.

(ii) If we write

$$\begin{aligned} 2^n &= (1+1)^n \\ &= 1^n + {}^nC_1 1^{n-1} + {}^nC_2 1^{n-2} 1^2 + \cdots + 1^n, \end{aligned}$$

we see that

$$\begin{aligned} 2^n &= 1 + n + \frac{n(n-1)}{2} + \cdots + 1^n \\ &= n + \text{positive terms}. \end{aligned}$$

So $2^n - n$ is positive, and hence $2^n > n$ for any positive integer n.

(iii) (a) Since n is a positive integer we have, using part (ii), that

$$0 < n < 2^n$$

and hence, by part (i), that

$$0 < \frac{1}{2^n} < \frac{1}{n}.$$

(b) As n gets larger, $\frac{1}{n}$ gets close to zero, and also $\frac{1}{2^n}$ is squeezed between zero and $\frac{1}{n}$. Since

$$\lim_{n\to\infty}\frac{1}{n}=0,$$

we must conclude that $\lim_{n\to\infty}\frac{1}{2^n}=0$.

Parts (ii) and (iii) justify assertions about the graph of $y = 2^x$ that we made in *Unit 2* of this block, since they show that $\lim_{x\to\infty} 2^x = \infty$, i.e. 2^x tends to infinity as x tends to infinity, and that $\lim_{x\to-\infty} 2^x = 0$.

Solution 4.5.2

(i) Since $s_n = a + ar + \cdots + ar^{n-1}$, we have

$$rs_n = \quad ar + ar^2 + \cdots + ar^n,$$

and so

$$s_n - rs_n = a - ar^n,$$

all other terms cancelling.

Hence $(1-r)s_n = a(1-r^n)$,

i.e. $s_n = \frac{a}{1-r}(1-r^n)$, provided $r \neq 1$.

(ii) If $-1 < r < 1$, then $\lim_{n\to\infty} r^n = 0$. All the other terms in the expression for s_n are independent of n. Thus

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}\times(1)$$

$$= \frac{a}{1-r}.$$

Solution 4.5.3

Since $0.111\ldots = \frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$, we have a geometric series with $a=\frac{1}{10}$, $r=\frac{1}{10}$. Since $r=\frac{1}{10}$ does satisfy $-1<r<1$, we have

$$0.111\ldots = \frac{a}{1-r}$$

$$= \frac{1/10}{1-1/10}$$

$$= \frac{1/10}{9/10}$$

$$= 1/9.$$

Solution 4.5.4

(i) At the first interest payment date, the bank adds

$\frac{1}{2}$ of 10% of sum in account,

i.e. £$(\frac{1}{2}\times 0.1\times 1)$.

The total in the account is now

$$£\left(1+\frac{0.1}{2}\right).$$

At the second payment date, the bank adds

$\frac{1}{2}\times 0.1\times$ current amount,

so that the total is

$$\left(1+\frac{0.1}{2}\right)\times \text{current amount},$$

i.e. $$£\left(1+\frac{0.1}{2}\right)\left(1+\frac{0.1}{2}\right) = £\left(1+\frac{0.1}{2}\right)^2$$

$$= £1.1025.$$

(You may feel that you arrived at this result more quickly. We set out the working as above because the method generalizes.)

(ii) This time, $\frac{0.1}{4}$ of the current value is added at each interest date, so the values are:

at the end of 1st quarter, £$\left(1+\frac{0.1}{4}\right)$;

at the end of 2nd quarter, £$\left(1+\frac{0.1}{4}\right)^2$; and so on.

The total at the end of the year will be

$$£\left(1+\frac{0.1}{4}\right)^4 \simeq £1.1038129.$$

(iii) Arguing as above, and using the y^x button on the calculator, the end-of-year values (in £) are:

$$\left(1+\frac{0.1}{5}\right)^5 \simeq 1.1040808;$$

$$\left(1+\frac{0.1}{6}\right)^6 \simeq 1.1042604;$$

$$\left(1+\frac{0.1}{7}\right)^7 \simeq 1.1043892;$$

etc.

With daily payments, we obtained

$$\left(1+\frac{0.1}{365}\right)^{365} \simeq 1.1051557.$$

For 10000 and 100000 payments a year, we obtained

$$\left(1+\frac{0.1}{10000}\right)^{10000} \simeq 1.1051698,$$

and $$\left(1+\frac{0.1}{100000}\right)^{100000} \simeq 1.1051709.$$

For larger numbers of payments we obtain 1.1051709.

Solution 4.5.5

For various values of n we obtained results as follows.

n	$\left(1+\frac{1}{n}\right)^n$ (approx.)
10	2.5937425
100	2.7048138
1 000	2.7169238
10 000	2.7181459
100 000	2.7182546
500 000	2.7182818
1 000 000	2.7182818

These results suggest that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \simeq 2.7182818.$$

Since e^1 (inverse ln 1) is approximately 2.7182818 (as found in *Unit 2* of this block), we might suspect that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$